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SOME INTEGRALS INVOLVING E-FUNCTIONS AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

By T. M. MACROBERT (*Glasgow*)

[Received 8 September 1941]

1. Introductory

In this paper some integrals involving products of simple cases of E-functions and related integrals of products of confluent hypergeometric functions are evaluated. In § 2 a list of known formulae required in the proofs is given. An integral involving one E-function is discussed in § 3; and the integrals involving products of the functions are then derived in § 4.

2. List of formulae

The E-function was defined in the *Proceedings of the Royal Society of Edinburgh*, 58 (1937), 3. Only simple cases of the function are employed in this paper, the following four formulae alone being required:

$$E(\alpha,\beta::x)=\Gamma(\alpha)\int\limits_0^\infty e^{-\lambda}\lambda^{\beta-1}\left(1+\frac{\lambda}{x}\right)^{-\alpha}d\lambda \quad (\mathrm{R}(\beta)>0), \tag{1}$$

$$E(\alpha,\beta::x)=\sum_{\alpha,\beta}\Gamma(\beta-\alpha)\Gamma(\alpha)x^{\alpha}{}_1F_1(\alpha;\alpha-\beta+1;x), \tag{2}$$

$$E(\tfrac{1}{2}-k-m,\tfrac{1}{2}-k+m::x)$$
$$=\Gamma(\tfrac{1}{2}-k-m)\Gamma(\tfrac{1}{2}-k+m)x^{-k}e^{\frac{1}{2}x}W_{k,m}(x), \tag{3}$$

$$\cos n\pi\, E(\tfrac{1}{2}+n,\tfrac{1}{2}-n::2x)=\sqrt{(2\pi x)}e^{x}K_n(x). \tag{4}$$

Note. In (2) the notation $\sum\limits_{\alpha,\beta}$ indicates that to the expression following the symbol is to be added a similar expression with α and β interchanged.

In addition, use will be made of the well-known formula for the hypergeometric function

$$F(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F\left(\begin{matrix}\alpha, & \beta;\\ \alpha+\beta-\gamma+1\end{matrix}\,1-x\right)+$$
$$+\frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-x)^{\gamma-\alpha-\beta}F\left(\begin{matrix}\gamma-\alpha, & \gamma-\beta;\\ \gamma-\alpha-\beta+1\end{matrix}\,1-x\right), \tag{5}$$

and of Dixon's theorem

$$
{}_3F_2\left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c \end{matrix} 1\right]
$$

$$
= \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}, \tag{6}
$$

where $R(2+a-2b-2c) > 0$.

3. An integral involving one E-function

If
$$
I \equiv \int_0^\infty e^{-t} t^{\gamma-1} E(\alpha, \beta :: tx)\, dt,
$$

where $x > 0$, $R(\alpha+\gamma) > 0$, $R(\beta+\gamma) > 0$, on substituting for the E-function from formula (1) and then putting $\lambda = \mu t$ and changing the order of integration it is found that

$$
I = \Gamma(\alpha) \int_0^\infty \mu^{\beta-1}\left(1+\frac{\mu}{x}\right)^{-\alpha} d\mu \int_0^\infty e^{-t(1+\mu)} t^{\beta+\gamma-1}\, dt
$$

$$
= \Gamma(\alpha)\Gamma(\beta+\gamma) \int_0^\infty \frac{\mu^{\beta-1}}{(1+\mu)^{\alpha+\beta+\gamma}}\left(1-\frac{\mu(x-1)}{(1+\mu)x}\right)^{-\alpha} d\mu.
$$

Hence, if $R(x) > \frac{1}{2}$, $R(\alpha+\gamma) > 0$, $R(\beta+\gamma) > 0$,

$$
\int_0^\infty e^{-t} t^{\gamma-1} E(\alpha, \beta :: tx)\, dt = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)} F\left(\begin{matrix} \alpha, & \beta; \\ \alpha+\beta+\gamma \end{matrix} \frac{x-1}{x}\right). \tag{7}
$$

This is equivalent to the formula

$$
\int_0^\infty e^{-t(\frac{1}{2}-x)} t^{l-1} W_{k,m}(t)\, dt
$$

$$
= \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)} F\left(\begin{matrix} l+m+\frac{1}{2}, & l-m+\frac{1}{2}; \\ l-k+1 \end{matrix} x\right) \tag{8}
$$

given by Goldstein.* To see this, replace $x, t, \alpha, \beta, \gamma$ in (7) by $1/(1-x)$, $(1-x)t$, $\frac{1}{2}-k-m$, $\frac{1}{2}-k+m$, $k+l$ respectively, and apply formula (3). A proof has also been given by Erdélyi.†

* Goldstein, *Proc. London Math. Soc.* (2) 34 (1932), 114.

† A. Erdélyi, *Quart. J. of Math.* (Oxford), 10 (1939), 189.

4. Integrals involving products of E-functions

On applying formula (2) to the first of the E-functions in the integral below and then integrating term by term, using formula (7), it is found that

$$\int_0^\infty e^{-t}t^{\gamma-1}E(\alpha,\beta::yt)E(\lambda,\mu::zt)\,dt$$

$$=\sum_{\alpha,\beta}\frac{\pi\Gamma(\lambda)\Gamma(\mu)}{\sin(\beta-\alpha)\pi}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\Gamma(\alpha+\gamma+\lambda+r)\Gamma(\alpha+\gamma+\mu+r)}{r!\,\Gamma(\alpha-\beta+1+r)\Gamma(\alpha+\gamma+\lambda+\mu+r)}\times$$

$$\times y^{\alpha+r}F\left(\begin{matrix}\lambda, & \mu; & \dfrac{z-1}{z}\\ \alpha+\gamma+\lambda+\mu+r & & \end{matrix}\right), \tag{9}$$

where $|y|<1$, $\mathrm{R}(z)>\frac{1}{2}$, $\mathrm{R}(\alpha+\gamma+\lambda)>0$, $\mathrm{R}(\beta+\gamma+\lambda)>0$, $\mathrm{R}(\alpha+\gamma+\mu)>0$, $\mathrm{R}(\beta+\gamma+\mu)>0$.

Now, when $y\to 1$, this becomes

$$\int_0^\infty e^{-t}t^{\gamma-1}E(\alpha,\beta::t)E(\lambda,\mu::zt)\,dt$$

$$=\sum_{\alpha,\beta}\frac{\pi\Gamma(\lambda)}{\sin(\beta-\alpha)\pi}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\Gamma(\alpha+\gamma+\mu+r)}{r!\,\Gamma(\alpha-\beta+1+r)}\times$$

$$\times\int_0^\infty\frac{v^{\mu-1}}{(1+v)^{\alpha+\gamma+\lambda+\mu+r}}\left\{1-\frac{v(z-1)}{(1+v)z}\right\}^{-\lambda}dv$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)}{\Gamma(\alpha+\beta+\gamma+\mu)}\Gamma(\alpha+\gamma+\mu)\Gamma(\beta+\gamma+\mu)\times$$

$$\times\int_0^\infty\frac{v^{\mu-1}}{(1+v)^{\gamma+\lambda+\mu}}\left\{1-\frac{v(z-1)}{(1+v)z}\right\}^{-\lambda}dv\times$$

$$\times\sum_{\alpha,\beta}\frac{\Gamma(\beta-\alpha)\Gamma(\alpha+\beta+\gamma+\mu)}{\Gamma(\beta+\gamma+\mu)\Gamma(\beta)}\frac{1}{(1+v)^{\alpha}}F\left(\begin{matrix}\alpha, & \alpha+\gamma+\mu; & \dfrac{1}{1+v}\\ & \alpha-\beta+1 & \end{matrix}\right)$$

But, by (5), the expression in the last line is equal to

$$\frac{1}{(1+v)^{\alpha}}F\left(\begin{matrix}\alpha, & \alpha+\gamma+\mu; & \dfrac{v}{1+v}\\ \alpha+\beta+\gamma+\mu & & \end{matrix}\right).$$

Hence, on making this substitution, expanding the hypergeometric series, and integrating term by term, we have

$$\int_0^\infty e^{-t}t^{\gamma-1}E(\alpha,\beta::t)E(\lambda,\mu::zt)\,dt$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)\Gamma(\alpha+\gamma+\lambda)\Gamma(\alpha+\gamma+\mu)\Gamma(\beta+\gamma+\mu)}{\Gamma(\alpha+\beta+\gamma+\mu)\Gamma(\alpha+\gamma+\lambda+\mu)}\times$$

$$\times\sum_{r=0}^{\infty}\frac{(\alpha)_r(\mu)_r(\alpha+\gamma+\mu)_r}{r!\,(\alpha+\beta+\gamma+\mu)_r(\alpha+\gamma+\lambda+\mu)_r}F\left(\begin{matrix}\lambda,\ \mu+r; \\ \alpha+\gamma+\lambda+\mu+r\end{matrix}\ \frac{z-1}{z}\right), \tag{10}$$

where $R(z) > \frac{1}{2}$, $R(\alpha+\gamma+\lambda) > 0$, $R(\beta+\gamma+\lambda) > 0$, $R(\alpha+\gamma+\mu) > 0$, $R(\beta+\gamma+\mu) > 0$.

By substituting from (3) and (4) in (9) and (10), integrals involving products of confluent hypergeometric functions and of modified Bessel functions of the second kind can be obtained.

For example, in (10) put $z = 1$ and replace α, β, λ, μ by $\frac{1}{2}+k+m$, $\frac{1}{2}+k-m$, $\frac{1}{2}-k-m$, $\frac{1}{2}-k+m$ respectively. Then, if $R(\gamma\pm 2m+1) > 0$,

$$\int_0^\infty t^{\gamma-1}W_{k,m}(t)W_{-k,m}(t)\,dt$$

$$= \frac{\Gamma(\gamma+1)\Gamma(1+2m+\gamma)\Gamma(\gamma+1)}{\Gamma(\frac{3}{2}+k+m+\gamma)\Gamma(\frac{3}{2}-k+m+\gamma)}\times$$

$$\times\,{}_3F_2\left[\begin{matrix}1+2m+\gamma,\ \frac{1}{2}+k+m,\ \frac{1}{2}-k+m; \\ \frac{3}{2}-k+m+\gamma,\ \frac{3}{2}+k+m+\gamma\end{matrix}\ 1\right].$$

But, by (6), the hypergeometric function is equal to

$$\frac{\Gamma(\frac{3}{2}+m+\frac{1}{2}\gamma)\Gamma(\frac{3}{2}-k+m+\gamma)\Gamma(\frac{3}{2}+k+m+\gamma)\Gamma(\frac{1}{2}-m+\frac{1}{2}\gamma)}{\Gamma(2+2m+\gamma)\Gamma(1-k+\frac{1}{2}\gamma)\Gamma(1+k+\frac{1}{2}\gamma)\Gamma(\gamma+1)}.$$

Therefore, if $R(\gamma\pm 2m+1) > 0$,

$$\int_0^\infty t^{\gamma-1}W_{k,m}(t)W_{-k,m}(t)\,dt = \frac{\Gamma(\gamma+1)\Gamma(\frac{1}{2}+m+\frac{1}{2}\gamma)\Gamma(\frac{1}{2}-m+\frac{1}{2}\gamma)}{2\Gamma(1+k+\frac{1}{2}\gamma)\Gamma(1-k+\frac{1}{2}\gamma)}. \tag{11}$$

ON BASIC DOUBLE HYPERGEOMETRIC FUNCTIONS

By F. H. JACKSON (*Eastbourne*)

[Received 14 January 1942: in revised form 29 April 1942]

1. Introduction

In recent papers by Burchnall and Chaundy* many most interesting and novel expansions of Appell's double hypergeometric functions are obtained and discussed. The expansions in question were obtained by using certain operative-functions, formed from hypergeometric functions in which the parameters were replaced by the partial-differential operators $x\partial/\partial x$, $y\partial/\partial y$. Mr. Chaundy suggested to the present writer that possibly a similar procedure using basic operators might give interesting expansions of new basic double hypergeometric functions. This has proved to be the case. As such functions have not been studied previously I give in this paper the q^θ, q^ϕ partial differential equations satisfied by the basic functions. These basic equations reduce in the limiting cases to the ordinary linear partial differential equations satisfied by Appell's functions. I do not consider it necessary to give detailed analysis in the case of every one of the many expansions which appear: in one or two typical cases only will there be detailed analysis, since the method and principles used are the same throughout the paper.

Basic functions are of increasing interest both in number theory and in combinatory analysis. This may be some justification for writing the present paper, which throughout is due to the suggestions and influence of Burchnall's and Chaundy's work. I wish here to acknowledge the kind help of Dr. W. N. Bailey in respect of Lemmas (24) and (25).

The Normal and Abnormal Functions

In the case of basic functions two different forms appear, which I term (i) Normal, (ii) Abnormal, respectively. The abnormality arises from the presence of a quadratic solitary factor q^{n^2} in all the terms of the series. The two kinds of series, though very similar in

* J. L. Burchnall and T. W. Chaundy, *Quart. J. of Math.* (Oxford), 11 (1940), 249–70.

appearance, differ widely owing to the presence of the quadratic solitary factor, which factor arises naturally, and, as will be seen, cannot be evaded.

2. The four normal functions

The functions

$$\Phi^{(1)}[a;b,b';c;x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n, \tag{1}$$

$$\Phi^{(2)}[a;b,b';c,c';x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(m)!\,(n)!\,(c)_m(c')_n} x^m y^n, \tag{2}$$

$$\Phi^{(3)}[a,a';b,b';c;x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n, \tag{3}$$

$$\Phi^{(4)}[a,b;c,c';x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(m)!\,(n)!\,(c)_m(c')_n} x^m y^n, \tag{4}$$

in which $\qquad (a)_r \equiv (1-q^a)(1-q^{a+1})\ldots(1-q^{a+r-1}),$

are basic analogues of functions discussed by Appell and Kampé de Fériet.*

3. The abnormal functions

The four abnormal functions differ from the four normal functions in the fact that

$$y^n q^{\frac{1}{2}n(n-1)}$$

replaces y^n in the general term and so in all terms of the series. The notation for the abnormal $\Phi^{(1)}$ will suffice to show the distinction.

$$\Phi^{(1)}[a;b,b';c;x,y;q] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n q^{\frac{1}{2}n(n-1)}. \tag{5}$$

The abnormality is indicated in the expression on the left by the presence of q among the elements.

In the limiting case when $|q| = 1$ the series $\Phi^{(1)}$, $\Phi^{(3)}$ converge absolutely when $|x|$, $|y| < 1$, $\Phi^{(2)}$ when $|x|+|y| < 1$, $\Phi^{(4)}$ when $|\sqrt{x}|+|\sqrt{y}| < 1$. It follows that for the same conditions there is even sharper convergence when $|q| \leqslant 1$, which condition will be assumed

* P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques* (Paris, 1926).

throughout the paper. It is obvious that in the case of the abnormal series the presence of the solitary quadratic factor in every term will ensure a very rapid and wide range of convergence.

4. Notation

As in previous papers by the writer,

$$\left.\begin{aligned}(\theta)_n &\equiv (1-q^{\theta})(1-q^{\theta+1})...(1-q^{\theta+n-1})\\ (\phi)_n &\equiv (1-q^{\phi})(1-q^{\phi+1})...(1-q^{\phi+n-1})\end{aligned}\right\}, \tag{6}$$

where $\theta,\ \phi \equiv x\partial/\partial x,\ y\partial/\partial y,\ q^{\theta} = \exp(\theta \log q),\ q^{\phi} = \exp(\phi \log q)$,

$$\{G(h)\}^{-1} = \lim_{\kappa\to\infty}(1-q^h)(1-q^{h+1})...(1-q^{h+\kappa}) \quad (|q|<1). \tag{7}$$

This function $\{G(h)\}^{-1}$ is not the basic gamma function,* but it serves to replace the ordinary gamma function in the basic formulae as used in the following analysis:

$$\nabla_q(h) \equiv \frac{G(h)G(h+\theta+\phi)}{G(h+\theta)G(h+\phi)}, \qquad \Delta_q(h) \equiv \frac{G(h+\theta)G(h+\phi)}{G(h)G(h+\theta+\phi)}. \tag{8}$$

The notation of Δ and its inverse ∇ due to Burchnall and Chaundy is of great convenience. It is by means of these two operators, and their equivalents in series, that the expansions of the basic double hypergeometric functions, which follow, are obtained. The question arises at once, Can such equivalences as

$$\nabla_q(h) = \sum_{r=0}^{\infty} \frac{(-\theta)_r(-\phi)_r}{(r)!\,(h)_r} q^{r(\theta+\phi+h)} \tag{9}$$

be justified? The answer is, Certainly they can. It is sufficient to note that the operations on $x^m y^n$ by $\nabla_q(h)$, $\Delta_q(h)$, $\nabla_q(h)\Delta_q(k)$, respectively and the operations by the respective series equivalent to the Δ operators, always produce terminating series and finite products, for example identities such as

$$\nabla_q(h)x^m y^n = \frac{(h)_{m+n}}{(h)_m(h)_n} x^m y^n \equiv \sum \frac{(-m)_r(-n)_r}{(r)!\,(h)_r} x^m y^n q^{r(h+m+n)}, \tag{10}$$

which is the basic form of Vandermonde's identity. Moreover, the infinite expansions obtained are absolutely convergent within the ranges stated for the variables x, y, and the base q.

* F. H. Jackson, *Proc. Royal Soc.* A, 76 (1905), 127–44; J. E. Littlewood, *Proc. London Math. Soc.* (2) 5 (1907), 395.

5. The fundamental transformations for the normal functions

Since $\nabla_q(h)$ transforms $(h)_m(h)_n x^m y^n$ into $(h)_{m+n} x^m y^n$, we can write

$$\left.\begin{aligned}
\Phi^{(2)}[a;b,b';c,c';x,y] &= \nabla_q(a)\Phi(a,b;c;x)\Phi(a,b';c';y)\\
\Phi^{(3)}[a,a';b,b';c;x,y] &= \Delta_q(c)\Phi(a,b;c;x)\Phi(a',b';c;y)\\
\Phi^{(1)}[a;b,b';c;x,y] &= \nabla_q(a)\Delta_q(c)\Phi(a,b;c;x)\Phi(a,b';c;y)\\
\Phi^{(1)}[a;b,b';c;x,y] &= \nabla_q(a)\Phi^{(3)}[a,a;b,b';c;x,y]\\
\Phi^{(1)}[a;b,b';c;x,y] &= \Delta_q(c)\Phi^{(2)}[a;b,b';c,c;x,y]\\
\Phi^{(4)}[a,b;c,c';x,y] &= \nabla_q(b)\Phi^{(2)}[a;b,b;c,c';x,y]
\end{aligned}\right\}. \tag{11}$$

From these transformations twelve interesting expansions of the above functions will be obtained by the use of certain theorems stated as lemmas in § 7 below.

6. The fundamental transformations for the abnormal functions

When we seek a basic analogue for the ordinary hypergeometric function $F(a,b;c;x+y)$ an abnormality appears, which of necessity introduces another type of function. I term this *abnormal*. The $(x+y)^n$ of the ordinary binomial expansion finds its parallel in basic algebra in a product

$$[x+y]_n \equiv (x+y)(x+yq)...(x+yq^{n-1}).$$

This is the basic binomial for positive integer n. Its well-known expansion is

$$\sum \frac{(n)!}{(r)!\,(n-r)!}\, x^{n-r}y^r q^{\frac{1}{2}r(r-1)}, \tag{12}$$

introducing the quadratic solitary factor, so that we have the abnormal function

$$\begin{aligned}
\Phi(a,b;c;[x+y]) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(m)!\,(n)!\,(c)_{m+n}}\, x^m y^n q^{\frac{1}{2}n(n-1)}\\
&= \sum_{N=0}^{\infty} \frac{(a)_N(b)_N}{(N)!\,(c)_N}(x+y)(x+yq)...(x+yq^{n-1}).
\end{aligned} \tag{13}$$

From these by means of the lemmas in § 7 below we obtain

$$\left.\begin{aligned}
\Phi(a,b;c;[x+y]) &= \nabla_q(b)\Phi^{(1)}[a;b,b;c;x,y;q]\\
\Phi(a,b;c;[x+y]) &= \Delta_q(c)\Phi^{(4)}[a,b;c,c;x,y;q]\\
\Phi(a,b;c;[x+y]) &= \nabla_q(b)\Delta_q(c)\Phi^{(2)}[a;b,b;c,c;x,y;q]
\end{aligned}\right\}. \tag{14}$$

The presence of the element q in the functions on the right denotes that a quadratic solitary factor $q^{\frac{1}{2}r(r-1)}$ is associated with y^r in the general term, and, of course, in every term of the infinite series.

7. Lemmas

Since $(a)_r \equiv (1-q^a)(1-q^{a+1})...(1-q^{a+r-1}) \equiv (a)(a+1)...(a+r-1)$, so that throughout the paper the terms of factorials are always basic elements—thus $(a+r) \equiv 1-q^{a+r}$—we shall have, in the case of an operative factorial $(-\theta)_r$,

$$(-\theta)_r x^m = (-m)(-m+1)...(-m+r-1)x^m$$
$$\equiv (-m)_r x^m, \tag{15}$$

which vanishes identically for all positive integral $m < r$.

Moreover, when $m = r$, we can, by a transformation, have also

$$(-m)_r \equiv (-)^r (r)!\, q^{-\frac{1}{2}r(r-1)}, \tag{16}$$

and so, by introducing the operator ϕ, we obtain

$$(-\theta)_r(-\phi)_r x^m y^n = (-m)_r(-n)_r x^m y^n, \tag{17}$$

which vanishes identically for $\min(m,n) < r$. This and the following three operations are of the first importance; the usefulness of the Δ_q operative functions entirely depends upon them:

$$\{1/(1-a-\theta-\phi)_r\}x^m y^n = (-)^r \frac{x^m y^n}{(a+m+n-r)_r} q^{r(a+m+n)-\frac{1}{2}r(r-1)}, \tag{18}$$

$$\{1/(\theta+a)_r(\phi+a)_r\}x^m y^n = \{(a+m)_r(a+n)_r\}^{-1} x^m y^n, \tag{19}$$

$$\{(\theta+c)^{-1}\}\Phi(a,b;c;x) = (c)_r^{-1}\Phi(a,b;c+r;x). \tag{20}$$

We have, in addition to

$$\nabla_q(h) \equiv \frac{G(h)G(h+\theta+\phi)}{G(h+\theta)G(h+\phi)} = \sum_{r=0}^{\infty} \frac{(-\theta)_r(-\phi)_r}{(r)!\,(h)_r} q^{r(h+\theta+\phi)}, \tag{21}$$

its inverse

$$\Delta_q(h) = \sum_{r=0}^{\infty} \frac{(-\theta)_r(-\phi)_r}{(r)!\,(1-h-\theta-\phi)_r} q^r \tag{22}$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(h)_{2r}(-\theta)_r(-\phi)_r}{(r)!\,(h+r-1)_r(h+\theta)_r(h+\phi)_r} q^{\frac{1}{2}r(r-1)+r(k+\theta+\phi)}, \tag{23}$$

and

$$\nabla_q(h)\Delta_q(k) = \sum_{r=0}^{\infty} \frac{(k-h)_r(-\theta)_r(-\phi)_r}{(r)!\,(k+r-1)_r(k+\theta)_r(k+\phi)_r(h)_r} q^{\frac{1}{2}(r-1)+r(k+\theta+\phi)}$$

$$= \sum_{r=0}^{\infty} \frac{(h-k)_r(-\theta)_r(-\phi)_r}{(r)!\,(h)_r(1-k-\theta-\phi)_r} q^r. \tag{24}$$

Further,

$$
{}_3\Phi_2\left[\begin{matrix}(h),(-\theta),(-\phi)\\(a),(b)\end{matrix};q^{a+b-h+\theta+\phi}\right]
$$

$$
=\frac{G(a)G(a+\theta+\phi)}{G(a+\theta)G(a+\phi)}\,{}_3\Phi_2\left[\begin{matrix}(b-h),(-\theta),(-\phi)\\(1-a-\theta-\phi),(b)\end{matrix};q^{h-a+1}\right]. \tag{25}
$$

The basic binomial identity is

If $$(a+b)_N \equiv (a+b)(a+bq)(a+bq^2)\dots(a+bq^{N-1}), \tag{26}$$

then $$\frac{(a+b)_N}{(N)!} \equiv \sum_{r+s=N} \frac{a^r b^s}{(r)!\,(s)!} q^{\frac{1}{2}-s(s-1)}. \tag{27}$$

It must be noted that the order r, s is important. A change to s, r would involve a change in the solitary index from s to r.

The basic analogue to Vandermonde's identity is

If $$[a]_N \equiv (1-q^a)(1-q^{a+1})\dots(1-q^{a+N-1}), \quad \textit{etc.}, \tag{28}$$

then $$\frac{[a+b]_N}{(N)!} \equiv \sum_{r+s=N} \frac{[a]_r[b]_s}{(r)!\,(s)!} q^{sa}. \tag{29}$$

Here also the order of r, s must be observed.*

8. The twelve normal expansions

The detailed analysis for one function will be sufficient to show how all the twelve normal expansions may be obtained.

From (11) we have

$$\Phi(a,b;c;x)\Phi(a',b';c;y) = \nabla_q(c)\Phi^{(3)}[a,a';b,b';c;x,y]. \tag{30}$$

Replace $\nabla_q(c)$ by its equivalent series (21), namely

$$\sum_{r=0}^{\infty} \frac{(-\theta)_r(-\phi)_r}{(r)!\,(c)_r} q^{r(c+\theta+\phi)}.$$

By operating on the general term of $\Phi^{(3)}$ we obtain

$$\frac{(-m)_r(-n)_r}{(r)!\,(c)_r} q^{r(c+m+n)} \frac{(a)_m(a')_n(b)_m(b')_n}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n. \tag{31}$$

Now $(-m)_r$ or $(-n)_r$ vanishes identically for $\min(m,n) < r$.

We therefore (for such value of m or $n = r$) replace all m, n by r;

* Heine, *Kugelfunctionen* (1878), 99; W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge, 1935), 69; G. N. Watson, *J. of London Math. Soc.* 4 (1929), 4–9; F. H. Jackson, *Quart. J. of Math.* (Oxford), 11 (1940), 16 (V).

then making use of transformation (3) we must form the proper solitary factor by putting $m = r$, $n = r$ in the expression

$$q^{\frac{1}{2}r(-2m+r-1)+\frac{1}{2}r(-2n+r-1)+r(c+m+n)}.$$

So we have
$$\frac{(a)_r(a')_r(b)_r(b')_r}{(r)!\,(c)_r(c)_{2r}}\,q^{r(c+r-1)}x^r y^r,$$

and consequently an expansion

$$\Phi(a,b;c;x)\Phi(a',b';c;y) = \sum\left\{\frac{(a)_r(a')_r(b)_r(b')_r}{(r)!\,(c)_r(c)_{2r}}x^r y^r q^{r(c+r-1)}\times\right.$$
$$\left.\times\Phi^{(3)}[a+r,a'+r;b+r,b'+r;c+2r;x,y]\right\}. \quad (32)$$

As similar methods only are needed in order to obtain other expansions, it is necessary merely to state the expansions, and in doing so, I follow the order of those in Burchnall and Chaundy's paper.* Each expansion is followed by its inverse.

$$\Phi^{(2)}[a;b,b';c,c';x,y] = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r(b')_r}{(r)!\,(c)_r(c')_r}x^r y^r q^{r(a+r-1)}\times$$
$$\times\Phi(a+r,b+r;c+r;x)\Phi(a+r,b'+r;c'+r;y), \quad (33)$$

$$\Phi(a,b;c;x)\Phi(a,b';c';y) = \sum_{r=0}^{\infty}(-)^r\frac{(a)_r(b)_r(b')_r}{(r)!\,(c)_r(c')_r}x^r y^r q^{ar+\frac{1}{2}r(r-1)}\times$$
$$\times\Phi^{(2)}[a+r;b+r,b'+r;c+r,c'+r;x,y], \quad (34)$$

$$\Phi^{(3)}[a,a';b,b';c;xy] = \sum_{r=0}^{\infty}(-)^r\frac{(a)_r(a')_r(b)_r(b')_r}{(r)!\,(c+r-1)_r(c)_{2r}}x^r y^r q^{rc+\frac{1}{2}r(r-1)}\times$$
$$\times\Phi(a+r,b+r;c+2r;x)\Phi(a'+r,b'+r;c'+2r;y), \quad (35)$$

$$\Phi(a,b;c;x)\Phi(a',b';c;y) = \sum_{r=0}^{\infty}\frac{(a)_r(a')_r(b)_r(b')_r}{(r)!\,(c)_r(c)_{2r}}x^r y^r q^{r(c+r-1)}\times$$
$$\times\Phi^{(3)}[a+r,a'+r;b+r,b'+r;c+2r;x,y], \quad (36)$$

$$\Phi^{(1)}[a;b,b';c;x,y] = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r(b')_r(c-a)_r}{(r)!\,(c+r-1)_r(c)_{2r}}x^r y^r q^{rc+\frac{1}{2}r(r-1)}\times$$
$$\times\Phi(a+r,b+r;c+2r;x)\Phi(a+r,b'+r;c+2r;y), \quad (37)$$

* J. L. Burchnall and T. W. Chaundy, *Quart. J. of Math.* (Oxford), 11 (1940), 253 et seq.

$$\Phi(a,b;c;x)\Phi(a,b';c;y) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(b')_r(c-a)_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{ar+\frac{1}{2}r(r-1)} \times$$
$$\times \Phi^{(1)}[a+r; b+r, b'+r; c+2r; x, y], \quad (38)$$

$$\Phi^{(1)}[a;b,b';c;x,y] = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r}{(r)!\,(c)_{2r}} x^r y^r q^{r(a+r-1)} \times$$
$$\times \Phi^{(3)}[a+r, a+r; b+r, b'+r; c+2r; x, y], \quad (39)$$

$$\Phi^{(3)}[a,a;b,b';c;x,y] = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(b')_r}{(r)!\,(c)_{2r}} x^r y^r q^{ar+\frac{1}{2}r(r-1)} \times$$
$$\times \Phi^{(1)}[a+r; b+r, b'+r; c+2r; x, y], \quad (40)$$

$$\Phi^{(1)}[a;b,b';c;x,y] = \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_r(b')_r}{(r)!\,(c+r-1)_r(c)_{2r}} x^r y^r q^{rc+\frac{3}{2}r(r-1)} \times$$
$$\times \Phi^{(2)}[a+2r; b+r, b'+r; c+2r, c+2r; x, y], \quad (41)$$

$$\Phi^{(2)}[a;b,b';c,c;x,y] = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r(b')_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+\frac{3}{2}r(r-1)} \times$$
$$\times \Phi^{(1)}[a+2r; b+r, b'+r; c+2r; x, y], \quad (42)$$

$$\Phi^{(4)}[a,b;c,c';x,y] = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r}{(r)!\,(c)_r(c')_r} x^r y^r q^{r(b+r-1)} \times$$
$$\times \Phi^{(1)}[a+2r; b+r, b+r; c+r, c'+r; x, y], \quad (43)$$

$$\Phi^{(2)}[a;b,b;c,c';x,y] = \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_r}{(r)!\,(c)_r(c')_r} x^r y^r q^{r(b+r-1)} \times$$
$$\times \Phi^{(4)}[a+2r, b+r; c+r, c'+r; x, y]. \quad (44)$$

This completes the expansions in which all the series are of normal form.

9. The six expansions for abnormal series

By using the operators $\nabla_q(b)$, $\Delta_q(c)$, $\nabla_q(b)\Delta_q(c)$ and their series equivalents as given in (21)–(27) we can obtain the following six expansions. Each expansion is followed by its inverse.

$$\Phi(a,b;c;[x+y]) = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r}{(r)!\,(c)_{2r}} x^r y^r q^{r(b+r-1)} \times$$
$$\times \Phi^{(1)}[a+2r; b+r, b+r; c+2r; x, y; q], \quad (45)$$

$$\Phi^{(1)}[a;b,b;c;x,y;q] = \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_r}{(r)!\,(c)_{2r}} x^r y^r q^{rb+\frac{1}{2}r(r-1)} \times$$
$$\times \Phi(a+2r, b+r; c+2r; [x+y]), \quad (46)$$

$$\Phi(a,b;c;[x+y]) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_{2r}}{(r)!\,(c+r-1)_r(c)_{2r}} x^r y^r q^{rc+\frac{1}{2}r(r-1)} \times$$
$$\times \Phi^{(4)}[a+2r, b+2r; c+2r, c+2r; x, y; q], \quad (47)$$

$$\Phi^{(4)}[a,b;c,c;x,y;q] = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_{2r}}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+\frac{1}{2}r(r-1)} \times$$
$$\times \Phi(a+2r, b+2r; c+2r; [x+y]), \quad (48)$$

$$\Phi(a,b;c;[x+y]) = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r(c-b)_r}{(r)!\,(c+r-1)_r(c)_{2r}} x^r y^r q^{rc+\frac{1}{2}r(r-1)} \times$$
$$\times \Phi^{(2)}[a+2r; b+r, b+r; c+2r, c+2r; x, y; q], \quad (49)$$

$$\Phi^{(2)}[a;b,b;c,c;x,y;q] = \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_r(c-b)_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{r(b+r-1)} \times$$
$$\times \Phi(a+2r, b+r; c+2r; [x+y]). \quad (50)$$

It is to be noted that, where the element q follows x, y in the functions $\Phi^{(1)}$, $\Phi^{(2)}$, etc., the functions are abnormal. A quadratic solitary factor appears in each term. In the general term of such functions we always have $x^r y^s$ multiplied by $q^{\frac{1}{2}s(s-1)}$.

10. Two exceptional expansions

Before giving these expansions, it is necessary to introduce a basic analogue of Taylor's theorem due to the present writer.*

If $f[x+h]$ be an absolutely convergent series

$$\sum_{r=0}^{\infty} A_r[x+h]_r,$$

in which $\quad [x+h]_r \equiv (x+h)(x+hq)(x+hq^2)...(x+hq^{r-1}),$

then
$$f[x+h] = \sum_{r=0}^{\infty} q^{\frac{1}{2}r(r-1)} \frac{h^r}{(r)!} \{D^r f(x)\}, \quad (51)$$

in which D denotes the basic operator $\dfrac{1-q^\theta}{x(1-q)} \equiv \dfrac{(\theta)}{x(1-q)}$. This operator is the basic analogue of d/dx and is such that

$$x^r\{D\}^r = (\theta)(\theta-1)...(\theta-r+1). \quad (52)$$

* F. H. Jackson, *Messenger of Math.* 39 (1910), 26–8.

If we apply this general theorem to the special basic hypergeometric function

$$ {}_2\Phi_1(a,b;c;[x+h]) $$

and replace the arguments x, h by (i) $x+y$, $-xy$; (ii) $x+y-xy$, $+xy$; we have the following two expansions:

$$\Phi(a,b;c;[x+y\mid -xy]) = \sum_{r=0}^{\infty}(-)^r\frac{(a)_r(b)_r}{(r)!\,(c)_r}x^ry^rq^{\frac{1}{2}r(r-1)}\times$$
$$\times\Phi(a+r,b+r;c+r;x+y), \quad (53)$$

$$\Phi(a,b;c;[x+y-xy\mid +xy]) = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(r)!\,(c)_r}x^ry^rq^{\frac{1}{2}r(r-1)}\times$$
$$\times\Phi(a+r,b+r;c+r;x+y-xy). \quad (54)$$

I term these *exceptional* expansions. They are basic analogues of Burchnall's and Chaundy's results (44), (45). The unusual notation $[x+y\mid -xy]$ is necessary to avoid confusion with $x+y-xy$: thus

$$[x+y\mid -xy]_r$$
$$\equiv (x+y-xy)(x+y-xyq)(x+y-xyq^2)...(x+y-xyq^{r-1}),$$
$$[x+y-xy\mid +xy]_r$$
$$\equiv (x+y-xy+xy)(x+y-xy+xyq)...(x+y-xy+xyq^{r-1}),$$

showing that $[x+y-xy\mid +xy]$ is not $(x+y)$ except when $|q|=1$.

Moreover, in

$$\Phi(a+r,b+r;c+r;x+y) \quad \text{and} \quad \Phi(a+r,b+r;c+r;x+y-xy),$$

we have ordinary basic hypergeometric series in which the variable $x+y$ appears in the general term as an ordinary binomial $(x+y)^n$ and $x+y-xy$ appears as a trinomial $(x+y-xy)^n$. Such series as these, in which we have factorials of basic form together with ordinary binomial forms, make it impossible to obtain basic analogues of Burchnall's and Chaundy's expansions (46)–(51). I therefore give the name *exceptional* to the series to avoid confusion with abnormal series.

I propose to consider generalization of Burchnall's and Chaundy's expansions (52)–(55) in a second paper dealing also with the basic confluent functions.

11. Duplication formula

The following special case of the expansion (37) seems interesting in connexion with Vandermonde forms.

If in $$\Phi^{(1)}[a;b,b';c;x,y]$$

we replace y by xq^b and apply the basic Vandermonde theorem, $\Phi^{(1)}$ reduces to a basic hypergeometric series $\Phi(a,b+b';c;x)$, and the theorem (37) becomes

$$\Phi(a,b+b';c;x) = \sum_{r=0}^{\infty} \frac{(c-a)_r(a)_r(b)_r(b')_r}{(r)!\,(c+r-1)_r(c)_{2r}} x^{2r}q^{r(a+b)+\frac{1}{2}r(r-1)}\times$$
$$\times\Phi(a+r,b+r;c+2r;x)\Phi(a+r,b'+r,c+2r,xq^b), \tag{55}$$

which is a quasi-addition formula for a pair of numerator elements.

In the same way we can obtain an inverse formula using the expansion (38):

$$\Phi(a,b;c;x)\Phi(a,b';c;xq^b) = \sum_{r=0}^{\infty} (-)^r \frac{(c-a)_r(b)_r(b')_r(a)_r}{(r)!\,(c)_r(c)_{2r}} x^{2r}q^{r(a+b)+\frac{1}{2}r(r-1)}\times$$
$$\times\Phi(a+r,b+b'+2r;c+2r,x). \tag{56}$$

From this, by giving special values to the parameters, we obtain q-generalizations of certain theorems first given by Bailey.*

In (56) put $c = b+b'$; then we have

$$\Phi(a,b;b+b';x)\Phi(a,b';b+b';xq^b)$$
$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(b')_r(b+b'-a)_r}{(r)!\,(b+b')_r(b+b')_{2r}} x^{2r}q^{r(a+b)+\frac{1}{2}r(r-1)}\times$$
$$\times\left\{1-\frac{(a+r)}{(1)}+\frac{(a+r)_2}{(2)!}-\ldots\right\}, \tag{57}$$

which, by well-known expansions and properties of factorials, gives us

$$\Phi(a,b;b+b';x)\Phi(a,b';b+b';xq^b)$$
$$= (1-xq^a)_{-a} \sum \frac{(a)_r(b)_r(b')_r(b+b'-a)_r}{(r)!\,(b+b')_r(b+b')_{2r}} x^{2r}(1-x)_{-r}\, q^{r(a+b)+\frac{1}{2}r(r-1)}. \tag{58}$$

These reduce to Bailey's theorem in the limiting case $|q| = 1$.

12. The q^θ, q^ϕ partial differential equations

The basic analogues of p, q, r, s, t which are generally used to represent

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x\partial y}, \frac{\partial^2}{\partial y^2}$$

are represented by P, Q, R, S, T respectively, where

$$P \equiv \frac{1-q^\theta}{x(1-q)}, \quad Q = \frac{1-q^\phi}{y(1-q)}, \quad R = P^2, \quad S = PQ, \quad T = Q^2.$$

* W. N. Bailey, *Proc. London Math. Soc.* (2) 38 (1935), 377–84.

For our function $\Phi^{(1)}$ we have

$$[(a)(b)+y(b)q^{a+\theta}Q+\{q^b(a+1)x+q^a(b)x-(c)\}P+$$
$$+\{xyq^{a+b+1}-yq^{c+\theta-1}\}S+x\{q^{a+b+\phi+1}-q^c\}R]\Phi^{(1)}=0,$$
$$[(a)(b')+x(b')q^{a+\phi}P+\{q^{b'}(a+1)y+q^a(b')y-(c)\}Q+$$
$$+\{xy^{a+b'+1}-xq^{c+\phi-1}\}S+y\{q^{a+b'+\theta+1}-q^c\}T]\Phi^{(1)}=0.$$

In factorial forms the above two equations become

$$[y(\theta+\phi+a)(\phi+b)-(\phi)(\theta+\phi+c'-1)]\Phi^{(1)}=0,$$
$$[x(\theta+\phi+a)(\theta+b')-(\theta)(\theta+\phi+c-1)]\Phi^{(1)}=0.$$

The equations for $\Phi^{(2)}$ are rather more simple. They are

$$[(a)(b)+y(b)q^{a+\theta}Q+\{q^a(b+1)x+q^b(a)x-(c)\}P+$$
$$+xyq^{a+b+1}S+x(xq^{a+b+1+\phi}-q^c)R]\Phi^{(2)}=0,$$
$$[(a)(b')+x(b')q^{a+\phi}P+\{q^a(b'+1)y+q^{b'}(a)y-(c)\}Q+$$
$$+xyq^{a+b'+1}S+y(yq^{a+b'+1+\theta}-q^c)T]\Phi^{(2)}=0.$$

The equations for $\Phi^{(3)}$ are

$$[(a)(b)+\{q^a(b+1)x+q^b(a)x-(c)\}P-yq^cS-x\{q^{a+b+1}x-q^c\}R]\Phi^{(3)}=0,$$
$$[(a')(b')+\{q^{a'}+(b'+1)y+q^{b'}(a')y-(c)\}Q-$$
$$-xq^cS-y\{q^{a'+b'+1}-q^c\}T]\Phi^{(3)}=0.$$

The equations for $\Phi^{(4)}$ are

$$[(a)(b)+\{q^a(b+1)x+q^b(a)x-(c)\}P+$$
$$+\{q^a(b+1)y+q^b(a)y\}q^\phi Q+q^{a+b}xy\{q^{\theta+\phi}+q\}S+$$
$$+y^2q^{a+b+\theta+1}T+x\{xq^{a+b+\phi+1}-q^c\}R]\Phi^{(4)}=0$$

and a second equation formed by interchanging θ, ϕ and x, y and by replacing c by c'. I need not write down the equation. The equations for $\Phi^{(3)}$ are the most simple. All the equations reduce in the limiting case $|q|=1$ to the partial differential equations for Appell's functions as given in Bailey's Tract on generalized hypergeometric series. I do not propose to discuss these equations in this paper, though it is obvious many results of interest may be deduced from them.

13. A basic integral for $\Phi^{(1)}$

I use the notation $S\,d(qx)$ to denote the operation which reverses

$$\Delta_x \equiv \frac{1-q^\theta}{x(1-q)}.$$

In former papers* on q-definite integrals I showed that

$$\underset{0}{S}\, u^{b-1}(1-qu)_{c-b-1}(1-q^a ux)_{-a}\, d(qu) = \frac{\Gamma_q(b)\Gamma_q(c-b)}{\Gamma_q(c)}\Phi(a,b;c;x),$$

$$\overset{1}{\underset{0}{S}}\,\overset{1}{\underset{0}{S}}\, u^{b-1}(1-qu)_{d-b-1}v^{c-1}(1-qv)_{e-c-1}(1-q^a uvx)_{-a}\, d(qu)d(qv)$$
$$= \frac{\Gamma_q(c)\Gamma_q(e-c)\Gamma_q(b)\Gamma_q(d-b)}{\Gamma_q(e)\Gamma_q(d)}\,{}_3\Phi_2(a,b;c;d,e;x)$$

in which Γ_q represents the basic gamma function.

By methods similar to those in the papers to which I refer we can show that

$$\frac{\Gamma_q(a)\Gamma_q(c-a)}{\Gamma_q(c)}\Phi^{(1)}[a;b;b';c;x,y]$$
$$= \overset{1}{\underset{0}{S}}\, u^{a-1}(1-qu)_{c-a-1}(1-q^b xu)_{-b}(1-q^{b'}xu)_{-b'}\, d(qu').$$

It is not possible to find similar basic integrals for the other functions $\Phi^{(2)}$, $\Phi^{(3)}$, $\Phi^{(4)}$, since trinomials appear and in basic notation there are no simple analogues for multinomial theorems. In the above formulae

$$(1-q^a\lambda)_{-a} = 1+\frac{(1-q^a)}{(1)}\lambda+\frac{(1-q^a)(1-q^{a+1})}{(2)!}\lambda^2+\ldots,$$

supposed absolutely convergent.

Also the basic gamma function has the property

$$\Gamma_q(n+1) = (1-q^n)\Gamma_q(n).$$

For a full discussion of this function I refer again to the papers by Littlewood and the present writer to which reference was given earlier in this paper. In conclusion, it is obvious that the subject offers great field for development in the case of functions with more parameters, and in discussion of confluent cases analogous to those given by Burchnall and Chaundy. I leave this, however, to what I hope may be a second paper.

* F. H. Jackson, *Quart. J. of Math.* 4 (1910), 193–203; 199; *American J. of Math.* 32 (1910), 305–14.

Conditions for Convergence

In order that the expansions may be absolutely convergent in the threefold multiple series which appear, conditions must be stated. I avail myself of the results given in great detail by Burchnall and Chaundy* for their series which are limiting cases when $|q| = 1$ of those of this paper. It is obvious that the same conditions will cover the series discussed in this paper not only in the limiting case $|q| = 1$ but under the condition $|q| < 1$, which condition I impose. The presence, moreover, of the quadratic solitary factor will ensure very rapid convergence.

* Loc. cit. 263–9.

A THEOREM ON INDEPENDENCE RELATIONS

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1. P. Hall* considered the following problem: *Given a finite number of abstract sets $A_1, A_2,..., A_n$, under what circumstances is it possible to select one element a_ν from each set A_ν in such a way that $a_1, a_2,..., a_n$ are different from each other?* He found that the following conditions which are obviously necessary are also sufficient for the possibility of this choice: *However one selects any number k of the sets A_ν, say $A_{\nu_1}, A_{\nu_2},..., A_{\nu_k}$, where $1 \leqslant k \leqslant n$; $1 \leqslant \nu_1 < \nu_2 < ... < \nu_k \leqslant n$, the sets thus selected contain between them at least k different elements.*†

The purpose of this note is to investigate what additional restrictions apart from their being distinct may be imposed upon the 'representatives' a_ν if the obviously necessary conditions for their existence are to be at the same time sufficient conditions. In particular, it will be shown that in the case when the elements of the sets A_ν are vectors in some euclidean space, we may postulate that the selected elements a_ν are linearly independent. This leads to

THEOREM 1. *Suppose that $A_1, A_2,..., A_n$ are sets whose elements are vectors in some euclidean space. Then, if no group of k of the sets A_ν is contained in a sub-space of $k-1$ dimensions ($k = 1, 2,..., n$), it is possible to select one vector a_ν from each set A_ν in such a way that $a_1, a_2,..., a_n$ are linearly independent.*

As a second example consider the case when the elements of the sets A_ν are polynomials in variables $t_1,..., t_m$, with coefficients in some field F. We call polynomials $a_1,..., a_k$ *independent* if no polynomial $f(w_1,..., w_k)$ exists with coefficients in F not all zero which has the property that $f(a_1,..., a_k) = 0$, identically in $t_1,..., t_m$.

THEOREM 2. *Suppose that $A_1,..., A_n$ are sets of polynomials. If any k of the sets A_ν contain between them at least k independent poly-*

* P. Hall, *J. of London Math. Soc.*, 10 (1934), 26–30.

† The 2^n-1 conditions obtained in this way are independent of each other as is shown by the example:

$$A_1 = A_2 = ... = A_k = \{1, 2, ..., k-1\};\ A_{k+1} = ... = A_n = \{1,..., n\}$$

in which every one of Hall's conditions is satisfied except the one referring to $\nu_1 = 1;\ \nu_2 = 2;\ ...;\ \nu_k = k$.

nomials ($k = 1, 2, \ldots, n$) *then it is possible to select one polynomial* a_ν *from each set* A_ν *in such a way that* $a_1, \ldots, a_n$ *are independent.*

Theorem 2, and in fact, an analogous more general theorem concerning arbitrary functions instead of polynomials, follows from Theorem 1. For, under suitable regularity assumptions, the functions $f_1(t_1, \ldots, t_m)$, $f_2(t_1, \ldots, t_m), \ldots, f_k(t_1, \ldots, t_m)$ are independent if and only if the k vectors whose components are the elements forming the rows of the matrix $\left(\frac{\partial f_\kappa}{\partial t_\mu}\right)_{\kappa,\mu}$, are independent.

2. Let us now pass on to the general case. Suppose that the letters a, u, x, y, z as well as a', etc. denote elements of some abstract set S. $A_1, A_2, \ldots, B_1, B_2, \ldots$ denote sub-sets of S. Let I be a function which associates with every ordered finite system $x_1, x_2, \ldots, x_m$ of elements of S a number $I(x_1, \ldots, x_m)$ which is either zero or unity. For convenience we admit the value $m = 0$ in which case $I(x_1, \ldots, x_m)$ is written as $I(-)$, and we suppose throughout that $I(-) = 1$.

I is called an *independence relation*, $I(x_1, \ldots, x_m) = 1$ corresponding to the assertion that the elements $x_1, \ldots, x_m$ are independent, if

(i) I is *decreasing*, i.e.

$$I(x_1, \ldots, x_m) \geqslant I(x_1, \ldots, x_m, x_{m+1})$$

for every $m \geqslant 0$ and every $x_1 \ldots, x_{m+1}$. (By removing any elements from a set of independent elements another set of independent elements is obtained.)

(ii) I is *commutative*, i.e.

$$I(x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_m}) = I(x_1, x_2, \ldots, x_m)$$

whenever $\lambda_1, \ldots, \lambda_m$ is a rearrangement of $1, \ldots, m$. (Independence of elements is not affected by changing the order in which they follow each other.)

(iii) I is *non-reflexive*, i.e.

$$I(x, x) = 0$$

for every x. (Two independent elements are always different from each other.)

(iv) I is *distributive*, i.e.

$$I(x_1, \ldots, x_m) I(y_1, \ldots, y_{m+1}) \leqslant \sum_{\mu=1}^{m+1} I(x_1, \ldots, x_m, y_\mu)$$

for every $m \geqslant 0$ and every $x_1, \ldots, x_m, y_1, \ldots, y_{m+1}$. (If m elements

$x_1,\ldots, x_m$ are independent and every one of $m+1$ elements $y_1,\ldots, y_{m+1}$ depends on the x_μ, then $y_1,\ldots, y_{m+1}$ are not independent.)

A system of sets $A_1,\ldots, A_n$ is said to have the *property H* (Hall's condition) with respect to I if, given any numbers k, $\nu_1,\ldots, \nu_k$ satisfying

$$1 \leqslant k \leqslant n, \qquad 1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_k \leqslant n,$$

the sum-set $A_{\nu_1}+A_{\nu_2}+\ldots+A_{\nu_k}$ contains elements $x_1,\ldots, x_k$ for which

$$I(x_1,\ldots,x_k) = 1.$$

Sets $A_1,\ldots, A_n$ are said to have the *property R* (existence of independent representatives) with respect to I if an element a_ν can be chosen from every set A_ν in such a way that $I(a_1,\ldots, a_n) = 1$.

The function $I = I_0$ defined by putting

$$I_0(x_1,\ldots,x_m) = 1$$

if and only if $x_1,\ldots, x_m$ are distinct, has properties (i), (ii), (iii), (iv). Hall's theorem, quoted at the beginning, asserts that H and R are equivalent with respect to I_0. This is a special case of

THEOREM 3. *With respect to every independence relation I, the properties H and R are equivalent.*

THEOREM 4. *If, with respect to a non-reflexive function I, the properties H and R are equivalent, then I is an independence relation.*

The author has not been able to decide which property should be substituted for non-reflexivity in order to make the correspondence between Theorem 3 and Theorem 4 a perfect one. It is obvious that there are functions I, decreasing, commutative, and distributive, but not non-reflexive, with respect to which H and R are equivalent, e.g. the trivial function defined by putting

$$I(x_1,\ldots,x_m) = 1$$

for every $m \geqslant 0$ and every choice of elements $x_1,\ldots, x_m$. On the other hand, an example will be given at the end of this note of a decreasing, commutative, and distributive function with respect to which H and R are not equivalent.

Familiar facts concerning linear relations between vectors show that Theorem 1 is a special case of Theorem 3, and it has already been pointed out how Theorem 2 follows from Theorem 1. It does not seem possible to establish the foregoing results by an adaptation of Hall's proof of his special case, nor is there any connexion between

the present note and an earlier paper* which generalized Hall's theorem in a different direction.

3. We start by proving the almost trivial Theorem 4. Suppose that I is non-reflexive, and that H and R are equivalent. Then I is decreasing. For let $m > 0$; $I(x_1,..., x_{m+1}) = 1$. Then the sets

$$A_\mu = \{x_\mu\} \quad (1 \leqslant \mu \leqslant m+1)$$

have the property R and therefore H. From the definition of H it is clear that then the system $A_1,..., A_m$ satisfies H, and hence R. This last statement means that

$$I(x_1,..., x_m) = 1.$$

Thus, bearing in mind that I takes only the values 0 and 1, we find that

$$I(x'_1,..., x'_m) \geqslant I(x'_1,..., x'_m, x'_{m+1})$$

for any $m \geqslant 0$ and any $x'_1,..., x'_{m+1}$.

Next we show that I is commutative. Suppose that

$$I(x_1,..., x_m) = 1$$

for some $m > 0$ and some elements $x_1,..., x_m$. Then the system of sets $A_1,..., A_m$ where

$$A_\mu = \{x_\mu\}$$

satisfies R and therefore H. From the definition of H it follows that, for any permutation

$$\begin{pmatrix} 1 & 2 & \dots & m \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \end{pmatrix}$$

the system $A_{\lambda_1}, A_{\lambda_2},..., A_{\lambda_m}$ satisfies H, and therefore R. This is the same as saying that

$$I(x_{\lambda_1},..., x_{\lambda_m}) = 1.$$

Thus, for any $m > 0$ and any elements $x'_1,..., x'_m$,

$$I(x'_{\lambda_1},..., x'_{\lambda_m}) = I(x'_1,..., x'_m).$$

Finally, I is distributive. For let $m > 0$,

$$I(x_1,..., x_m) = I(y_1,..., y_{m+1}) = 1. \tag{1}$$

Put

$$A_\mu = \{x_\mu\} \quad (1 \leqslant \mu \leqslant m),$$

$$A_{m+1} = \{y_1,..., y_{m+1}\}.$$

Then $A_1,..., A_m$ satisfy R and therefore H. Now, since A_{m+1} contains $m+1$ 'independent' elements, a moment's consideration shows that $A_1,..., A_m, A_{m+1}$ satisfy H. Hence these sets satisfy R, i.e. for

* R. Rado, *Proc. London Math. Soc.* 44 (1938), 61–91.

at least one value of μ_0 $(1 \leqslant \mu_0 \leqslant m+1)$ we have $I(x_1,\ldots,x_m,y_{\mu_0}) = 1$. Then

$$\sum_{\mu=1}^{m+1} I(x_1,\ldots,x_m,y_\mu) \geqslant I(x_1,\ldots,x_m)I(y_1,\ldots,y_{m+1}). \tag{2}$$

In view of the fact that I takes only the values 0 and 1, we see that (2) holds for any $x_1,\ldots, x_m, y_1\ldots, y_{m+1}$, irrespective of whether (1) is true or not true.

4. We now proceed to prove the principal Theorem 3. Consider a decreasing, commutative, non-reflexive, and distributive function I with respect to which H and R are to be taken. The implication $R \to H$ is trivial. For, if $A_1,\ldots, A_n$ satisfy R, then every A_ν contains an element a_ν which is such that $I(a_1,\ldots,a_n) = 1$. If now $1 \leqslant k \leqslant n$; $1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_k \leqslant n$, then the set $A_{\nu_1}+A_{\nu_2}+\ldots+A_{\nu_k}$ contains the elements $a_{\nu_1}, a_{\nu_2},\ldots, a_{\nu_k}$, and we have

$$I(a_{\nu_1},\ldots,a_{\nu_k}) \geqslant I(a_1,\ldots,a_n) = 1.$$

Therefore $A_1,\ldots, A_n$ satisfy H.

On the other hand, let us suppose that the sets $A_1,\ldots, A_n$ have the property H. We have to prove that they satisfy R. If $n = 1$, then H and R are identical in meaning. We may therefore assume that $n > 1$, and that the equivalence of H and R has already been established for any system of $n-1$ sets $A_1',\ldots, A_{n-1}'$. Then there are elements x_ν in A_ν $(1 \leqslant \nu < n)$ satisfying

$$I(x_1,\ldots,x_{n-1}) = 1.$$

For, since $A_1,\ldots, A_n$ satisfy H, the same is true for $A_1,\ldots, A_{n-1}$, and then the existence of the x_ν follows from our induction hypothesis.

Furthermore, since $A_1,\ldots, A_n$ satisfy H, we can find elements $y_1,\ldots, y_n$ in the set $A_1+A_2+\ldots+A_n$ for which

$$I(y_1,\ldots,y_n) = 1.$$

The distributive inequality

$$1 = I(x_1,\ldots,x_{n-1})I(y_1,\ldots,y_n) \leqslant \sum_{\nu=1}^{n} I(x_1,\ldots,x_{n-1},y_\nu)$$

shows that for some index ν_0 $(1 \leqslant \nu_0 \leqslant n)$ we have

$$I(x_1,\ldots,x_{n-1},y_{\nu_0}) = 1.$$

Case I. Suppose that y_{ν_0} is contained in A_n.

Then $x_1,\ldots, x_{n-1}, y_{\nu_0}$ are in $A_1,\ldots, A_{n-1}, A_n$ respectively, and this last system of sets has the property R.

Case II. Suppose that y_{ν_0} belongs to A_{λ_0} where $1 \leqslant \lambda_0 < n$. Putting

$$B_1 = A_{\lambda_0}, \quad B_{\lambda_0} = A_1,$$
$$B_\nu = A_\nu \quad (1 \leqslant \nu \leqslant n;\ \nu \neq 1, \lambda_0),$$
$$z_1 = x_{\lambda_0}, \quad z_{\lambda_0} = x_1, \quad z_1' = y_{\nu_0},$$
$$z_\nu = x_\nu \quad (1 < \nu < n;\ \nu \neq \lambda_0),$$

we see that

$$z_\nu \text{ is in } B_\nu \quad (1 \leqslant \nu < n),$$
$$z_1' \text{ is in } B_1,$$
$$I(z_1', z_1, \ldots, z_{n-1}) = 1.$$

Furthermore, $B_1, \ldots, B_n$ satisfy H. In particular, $B_2, \ldots, B_n$ satisfy H, and hence, by induction, R. There are elements u_ν in B_ν $(1 < \nu \leqslant n)$ for which

$$I(u_2, \ldots, u_n) = 1.$$

Define $r(u_2, \ldots, u_n)$ as being the number of indices ν with $1 < \nu < n$ for which $u_\nu = z_\nu$. Then

$$0 \leqslant r(u_2, \ldots, u_n) \leqslant n-2.$$

Choose $u_2, \ldots, u_n$ in such a way that, for the $z_1, \ldots, z_{n-1}, z_1'$ under consideration, $r(u_2, \ldots, u_n)$ is maximal. We have

$$\left.\begin{aligned} 1 &= I(u_2, \ldots, u_n) I(z_1', z_1, \ldots, z_{n-1}) \\ &\leqslant I(u_2, \ldots, u_n, z_1') + \sum_{\nu=1}^{n-1} I(u_2, \ldots, u_n, z_\nu) \end{aligned}\right\}. \tag{3}$$

Therefore at least one term on the right-hand side of (3) is positive.

Case II'. Suppose that $I(u_2, \ldots, u_n, z_1') = 1$.

Then z_1', $u_2, \ldots, u_n$ are respectively in $B_1, B_2, \ldots, B_n$, and this last system has the property R. Hence $A_1, \ldots, A_n$ satisfies R.

Case II''. Suppose that $I(u_2, \ldots, u_n, z_1) = 1$.

Then precisely the same conclusion follows.

Case II'''. Suppose that, for some index ν_1 $(1 < \nu_1 < n)$,

$$I(u_2, \ldots, u_n, z_{\nu_1}) = 1.$$

Then

$$I(u_{\nu_1}, z_{\nu_1}) \geqslant I(u_2, \ldots, u_n, z_{\nu_1}),$$
$$u_{\nu_1} \neq z_{\nu_1},$$
$$I(u_2, \ldots, u_{\nu_1-1}, z_{\nu_1}, u_{\nu_1+1}, \ldots, u_n) = 1.$$

The elements $\quad u_2, \ldots, u_{\nu_1-1}, z_{\nu_1}, u_{\nu_1+1}, \ldots, u_n$

are respectively in $\quad B_2, \ldots, B_{\nu_1-1}, B_{\nu_1}, B_{\nu_1+1}, \ldots, B_n$,

and we have

$$r(u_2,\ldots, u_{\nu_1-1}, z_{\nu_1}, u_{\nu_1+1},\ldots, u_n) = r(u_2,\ldots, u_{\nu_1},\ldots, u_n)+1,$$

which contradicts the fact that $r(u_2,\ldots, u_n)$ has its largest possible value. This concludes our proof.

The example, referred to in §2 above, of a function I which is decreasing, commutative, and distributive, but with respect to which H and R are not equivalent is as follows. Put $S = \{1, 2, 3\}$. Define I by putting

$$I(-) = I(x) = I(x, y) = I(x, y, z) = 1$$

whenever the set $\{x, y, z\}$ contains exactly two different elements, and $I(x_1, x_2,\ldots, x_m) = 0$ in all other cases. It is easily verified that I has the required properties. If we put

$$A_1 = \{1\}, \qquad A_2 = \{2\}, \qquad A_3 = \{3\},$$

then the system A_1, A_2, A_3 satisfies, with respect to I, Hall's condition H. On the other hand, there are no independent representatives of A_1, A_2, A_3, i.e. R does not hold. I is, of course, not non-reflexive.

DIFFERENTIAL EQUATIONS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

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1. Introduction

In the course of a discussion with Mr. Chaundy regarding the 'three-term' differential equation

$$\{f(\delta)-x^r g(\delta)+x^{2r}h(\delta)\}y = 0 \quad (\delta = xd/dx) \tag{1}$$

it occurred to me to form the linear ordinary differential equations satisfied by certain functions $F(px, qx)$, where p, q are parameters, x is the variable of differentiation, and F is a hypergeometric function of two variables of a type specified in each case. Except in the final illustration, where I extend my method to Whittaker's functions, the equations obtained are of the type (1), and it seems worth while to record these instances in which an equation, so intractable in general, may be solved in terms of known functions. One point of special interest, discussed in detail later, may be mentioned here. If we take F to be a product of Bessel's functions of different orders and arguments, my method gives an equation of the fifth order instead of the anticipated equation of the fourth order. The former, though possessing an irrelevant solution, is in some respects a simpler equation than the latter, and the irrelevant solution has a simple and interesting form. I show in § 9 that a similar state of affairs exists when we consider products of functions which satisfy a large and important class of differential equations of the second order.

My method is based on the elementary observation that

$$p\frac{\partial}{\partial p}F(px,qx)+q\frac{\partial}{\partial q}F(px,qx) = x\frac{d}{dx}F(px,qx)$$

or, setting $p\dfrac{\partial}{\partial p} = \theta$, $q\dfrac{\partial}{\partial q} = \phi$, $x\dfrac{d}{dx} = \delta$, that

$$\theta+\phi = \delta, \tag{2}$$

for operations on the functions considered. In practically all the cases dealt with the function F satisfies two 'two-term' partial differential equations

$$f_1(\theta,\phi)F = pxg_1(\theta,\phi)F, \qquad f_2(\theta,\phi)F = qxg_2(\theta,\phi)F,$$

where f_r, g_r are polynomial in their arguments with coefficients independent of p, q, x. By suitable manipulation it is possible to obtain from these equations an equation in which θ, ϕ occur only in the association $\theta+\phi$. On replacing $\theta+\phi$ by δ we have an ordinary differential equation satisfied by F regarded as a function of x.

The connexion between systems of partial differential equations and ordinary equations may of course be explored in either direction, and I am indebted to a referee for drawing my attention to a series of papers by Horn, of which (4) is the latest, written from the opposite point of view to that taken here. Horn's aims and methods are markedly unlike those of the present paper and the points of contact between our work would seem to be slight.

We may illustrate the method and provide a starting-point for further development by considering the function

$$z = (1-px)^{-b}(1-qx)^{-b'} = {}_1F_0(b;px)\,{}_1F_0(b',qx),$$

where the notation is that usual for hypergeometric functions. Since $(1-x)^{-b}$ is a solution of the equation

$$\delta y = x(\delta+b)y,$$

it is evident that z satisfies the partial differential equations

$$\theta z = px(\theta+b)z, \qquad \phi z = qx(\phi+b')z. \tag{3}$$

From these, by addition and use of (2) and (3),

$$\begin{aligned}\delta z = (\theta+\phi)z &= px(\delta+b-\phi)z+qx(\delta+b'-\theta)z\\ &= x[p(\delta+b)+q(\delta+b')]z-pqx^2[\phi+b'+\theta+b]z.\end{aligned}$$

Thus z satisfies the ordinary equation

$$\delta z-x[p(\delta+b)+q(\delta+b')]z+pqx^2(\delta+b+b')z = 0. \tag{4}$$

Suppose now that the equations (3) are replaced by

$$\left.\begin{aligned}\theta f(\theta+\phi)z &= px(\theta+b)g(\theta+\phi)z\\ \phi f(\theta+\phi)z &= qx(\phi+b')g(\theta+\phi)z\end{aligned}\right\}. \tag{5}$$

Then, by a similar manipulation,

$$\delta f(\delta)z = px(\delta+b-\phi)g(\delta)z+qx(\delta+b'-\theta)g(\delta)z,$$

i.e.

$$\delta f(\delta)z-[px(\delta+b)+qx(\delta+b')]z+px\phi g(\delta)z+qx\theta g(\delta)z = 0. \tag{6}$$

Operate now on (6) with $f(\delta-1)$ and observe that

$$\begin{aligned} f(\delta-1)px\phi g(\delta)z &= pxf(\delta)\phi g(\delta)z \\ &= pxg(\delta)qx(\phi+b')g(\delta)z \\ &= pqx^2(\phi+b')g(\delta)g(\delta+1)z, \end{aligned}$$

with a similar result for the term in q. Thus, performing the operation stated on (6), we see that

a function z which satisfies the partial equations (5) *also satisfies the ordinary equation*

$$\delta f(\delta)f(\delta-1)z-x[p(\delta+b)+q(\delta+b')]f(\delta)g(\delta)z+ \\ +pqx^2(\delta+b+b')g(\delta)g(\delta+1)z = 0. \quad (7)$$

There is of course nothing in the argument which precludes the existence of solutions of (7) which are not solutions of (5).

2. The equations of $F^{(1)}[a;b,b';c;px,qx]$

The function

$$F^{(1)}[a;b,b';c;px,qx] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n(px)^m(qx)^n}{m!\,n!\,(c)_{m+n}}$$

satisfies the partial differential equations

$$\begin{aligned} \theta(\theta+\phi+c-1)z &= px(\theta+b)(\theta+\phi+a)z, \\ \phi(\theta+\phi+c-1)z &= qx(\phi+b')(\theta+\phi+a)z, \end{aligned} \quad (8)$$

which are of the form (5) with

$$f = \theta+\phi+c-1, \qquad g = \theta+\phi+a.$$

Thus, applying (7),

$F^{(1)}[a;b,b';c;px,qx]$ is a solution of the differential equation

$$\delta(\delta+c-1)(\delta+c-2)z-x[p(\delta+b)+q(\delta+b')](\delta+c-1)(\delta+a)z+ \\ +pqx^2(\delta+b+b')(\delta+a)(\delta+a+1)z = 0. \quad (9)$$

If in this we make the substitution $z = x^{1-c}z'$ and suppress the accent, we obtain the equation

$$\delta(\delta-1)(\delta+1-c)z-x[p(\delta+b+1-c)+q(\delta+b'+1-c)]\delta(\delta+a+1-c)z \\ +pqx^2(\delta+b+b'+1-c)(\delta+a+1-c)(\delta+a+2-c)z = 0. \quad (10)$$

The singular points of this equation are

$$x = 0,\ \infty,\ p^{-1},\ q^{-1}.$$

The substitutions

$$z_1 = x^{c-a-1}z,\ x_1 = x^{-1},\ p_1 = p^{-1},\ q_1 = q^{-1}$$

transform it into an equation identical in form with parameters

$$a_1 = b+b'+1-c,\ b_1 = b,\ b_1' = b',\ c_1 = b+b'+1-a,$$

while the change of variable $x_2 = p^{-1}-x$ also preserves the form of (10) with new parameters defined by

$$a_2 = 1-b,\ b_2 = 1-a,\ b_2' = b',\ c_2 = c+1-a-b,$$
$$p_2 = p,\ q_2 = pq(q-p)^{-1}.$$

Thus all the singularities have the same character, their exponents at

$$\text{being}\quad \begin{array}{cccc} 0, & \infty, & p^{-1}, & q^{-1} \\ \hline 0, & a-c+1, & 0 & 0 \\ 1, & a-c+2, & 1 & 1 \\ c-1, & b+b'+1-c, & c-a-b, & c-a-b'. \end{array} \tag{11}$$

We note that, in accordance with Fuchs's relation, the sum of the exponents is 6.

Since in (10) the operator δ is a factor of the middle as well as of the leading term, the unit difference of the exponents 0, 1 at the origin will not give rise to a logarithmic solution of the equation. There will indeed be two series solutions led respectively by a constant term and a multiple of x, or, alternatively, two independent solutions led by a constant term. Since all singularities of (10) have the same character, the equation will have no logarithmic solutions. This, evidently, is equally true of (9).

Before investigating the solutions of (9) valid near the origin we may remark that, if in (10) we set

$$p = 1,\ q = y^{-1},\ b+b'+1-c = \alpha,\ a+1-c = \beta,\ b' = \beta',\ 2+b'-c = \gamma,$$

we obtain the equation given by Appell* as that satisfied by $F^{(1)}[\alpha;\beta,\beta';\gamma;x,y]$ when regarded as a function of x only. The relevance of this result to the first formula of (13) below will be obvious.

3. Solutions valid near the origin

The table of functions $F^{(1)}$ given by Appell† furnishes sixty solutions of (9). Between any four there is a linear relation with coefficients independent of p, q, x, and there are, moreover, triads between

* (1), 75. † (1), 62–5.

which no such relation exists. There is thus a presumption that (9) can be completely solved in terms of functions $F^{(1)}$, and this is in general the case. We may indeed select from the table as solutions valid near the origin the following functions:

$$\begin{aligned} z_1 &= F^{(1)}(a;b,b';c;px,qx) \\ &= \sum_0^\infty \frac{(a)_n(b')_n}{n!\,(c)_n}(qx)^n F(-n,b;1-n-b';p/q) \\ &= \sum_0^\infty \frac{(a)_n(b+b')_n}{n!\,(c)_n}(qx)^n F(-n,b;b+b';(q-p)/q), \end{aligned} \tag{12}$$

where F is the ordinary hypergeometric function ${}_2F_1$, and the coefficient of x^n is a polynomial in p, q of degree n;*

$$\begin{aligned} z_{14} &= x^{1-c}F^{(1)}(b+b'+1-c;a+1-c,b';2+b'-c;px,p/q) \\ &= x^{1-c}\sum_{n=0}^\infty \frac{(b+b'+1-c)_n(a+1-c)_n}{n!\,(2+b'-c)_n}(px)^n\times \\ &\qquad \times F(b+b'+n+1-c,b';b'+n+2-c;p/q); \end{aligned} \tag{13}$$

and

$$\begin{aligned} z_{15} &= x^{1-c}F^{(1)}(b+b'+1-c;b,a+1-c;2+b-c;q/p,qx) \\ &= x^{1-c}\sum_{n=0}^\infty \frac{(b+b'+1-c)_n(a+1-c)_n}{n!\,(2+b-c)_n}(qx)^n\times \\ &\qquad \times F(b+b'+n+1-c,b;b+n+2-c,q/p). \end{aligned} \tag{14}$$

If in (14) we employ

$$\begin{aligned} &F(b+b'+n+1-c,b;b+n+2-c;q/p) \\ &\quad = (-p/q)^{b+b'+n+1-c}\frac{\Gamma(b+n+2-c)\Gamma(c-b'-n-1)}{\Gamma(1-b')\Gamma(b)}\times \\ &\qquad \times F(b+b'+n+1-c,b';2+b'-c;p/q)+ \\ &\qquad + (-p/q)^b\frac{\Gamma(b+n+2-c)\Gamma(b'+n+1-c)}{\Gamma(n+2-c)\Gamma(b+b'+n+1-c)}\times \\ &\qquad \times F(c-n-1,b;c-b'-n;p/q), \end{aligned} \tag{15}$$

we obtain a further solution

$$z'_{15} = x^{1-c}\sum_{n=0}^\infty \frac{(b'+1-c)_n(a+1-c)_n}{n!\,(2-c)_n}(qx)^n F(c-n-1,b;c-b'-n;p/q). \tag{16}$$

It is natural to adopt z_1, z_{14}, z'_{15} as appropriate solutions when

* Cf. (4), 450.

$|p/q| < 1$. Solutions appropriate when $|p/q| > 1$ may be obtained from these by permuting p with q and b with b'. We may add that the formulae connecting contiguous hypergeometric functions may be employed to verify directly that the power series in x occurring in (12)–(14) and (16) are solutions of (9).

4. Some special cases

We have seen that (9) may in general be solved in terms of functions $F^{(1)}$, but to this general statement I note two exceptions of some interest.

If $p = q$, which we may note in passing is a singular line of the system (8), we may without loss of generality assume $p = q = 1$. The point $x = 1$ is then a confluent singularity of (9) and (10): investigation shows it to be regular, and it remains true that solutions valid in its neighbourhood are free of logarithms. On inspecting the solutions obtained in § 3, we find that z_{14} and z_{15} are, save for a constant factor, identical. The equation (9) may now be written

$$[(\delta+c-2)-x(\delta+a)][\delta(\delta+c-1)-x(\delta+a)(\delta+b+b')]z = 0$$

with the obvious solutions*

$$z_\alpha = F(a, b+b'; c; x),$$
$$z_\beta = x^{1-c}F(b+b'+1-c, a+1-c; 2-c; x),$$

which are the degenerate forms of z_1, z_{14}.

Since $(\delta+c-2)y = x(\delta+a)y$ is satisfied by $y = x^{2-c}(1-x)^{c-a-2}$, any particular integral of

$$\delta(\delta+c-1)z - x(\delta+a)(\delta+b+b')z = Cx^{2-c}(1-x)^{c-a-2},$$

where C is any convenient constant, provides a third solution of (9), and, by variation of parameters, we obtain for this third solution

$$z = \int^{x} (1-t)^{b+b'-2}\{z_\alpha(t)z_\beta(x) - z_\alpha(x)z_\beta(t)\}\,dt.$$

Another case of failure is at first sight more difficult to understand. We observe that, if $a+1 = c$, the solutions z_{14}, z_{15}, z'_{15} all reduce to x^{1-c} and we require an additional solution of (9). It is slightly more convenient to consider the equivalent equation (10), which may now be written

$$(\delta-1)\{\delta+1-c-x[p(\delta+b+1-c)+q(\delta+b'+1-c)]+ \\ +pqx^2(\delta+b+b'+1-c)\}\delta z = 0. \quad (17)$$

* Cf. (1), 23, formula (25).

An obvious solution is $z = 1$: to obtain others write $\delta z = x^{c-1}u$, when (17) becomes

$$(\delta+c-2)\{\delta-x[p(\delta+b)+q(\delta+b')]+pqx^2(\delta+b+b')\}u = 0. \quad (18)$$

By (4) one solution of (18) is $u = (1-px)^{-b}(1-qx)^{-b'}$, leading to

$$z = \int x^{c-2}(1-px)^{-b}(1-qx)^{-b'}\,dx$$

$$= (c-1)^{-1}x^{c-1}F^{(1)}(c-1;b,b';c;px,qx), \quad (19)$$

i.e. back to an appropriately modified z_1 as a solution of (9). Elementary methods give as a second solution of (18)

$$u = (1-px)^{-b}(1-qx)^{-b'}\int x^{1-c}(1-px)^{b-1}(1-qx)^{b'-1}\,dx$$

leading to a series solution of (10) which does not appear to be hypergeometric.

To explain fully the reason for the present case of failure we must recall that the function $F^{(1)}$ satisfies not only (8) but also* the further equation

$$\Delta z \equiv q\theta(\phi+b')z-p\phi(\theta+b)z = 0, \quad (20)$$

which is stated by Appell* to be a consequence of (8). It is upon the system of *three* equations that his discussion of the sixty functions $F^{(1)}$ is based, and he has apparently failed to observe that the elimination he performs is impossible when $a+1 = c$. We may state the matter briefly thus. From (8) we may deduce

$$(\theta+\phi+c-2)\Delta z = 0 = (\theta+\phi+a-1)\Delta z.$$

Thus *either* $\Delta z = 0$ *or* $a+1 = c$, and in the latter case the equations (8) have the solution $z = (px)^{-a}\psi(p/q)$, where ψ is an arbitrary function. This is evidently not a solution of (20).

If in (8) we write

$$(\theta+\phi+c-1)z = (\theta+\phi+a)z = u,$$

then the equations for u take the form (3), and so, sufficiently,

$$(\delta+c-1)z = u = (1-px)^{-b}(1-qx)^{-b'},$$

whence, in agreement with (19),

$$z = x^{1-c}\int x^{c-2}(1-px)^{-b}(1-qx)^{-b'}\,dx.$$

Thus, if $a+1 = c$, the system (8) possesses only two solutions which

* (1), 53.

are independent functions of x, and the necessity for an additional solution of (9) or (10) is explained.

5. Extensions and particular cases

The hypergeometric function of two variables

$$F(px,qx) = \sum_0^\infty \sum_0^\infty \frac{(a_1)_{m+n}\cdots(a_r)_{m+n}(b)_m(b')_n}{(c_1)_{m+n}\cdots(c_s)_{m+n}\, m!\, n!}(px)^m(qx)^n,$$

where, for convergence, $s \geqslant r$, satisfies (5) with

$$f(\theta+\phi) = \prod_{t=1}^{s}(\theta+\phi+c_t-1), \qquad g(\theta+\phi) = \prod_{t=1}^{r}(\theta+\phi+a_t),$$

and is thus a solution of an ordinary equation of form (7) of order $2s+1$. A complete discussion of all the solutions would appear to require the construction of a table corresponding to Appell's table of $F^{(1)}$'s.

In particular the confluent function

$$\Phi_2(b,b';c;px,qx) = \sum_0^\infty \sum_0^\infty \frac{(b)_m(b')_n}{(c)_{m+n}\, m!\, n!}(px)^m(qx)^n$$

satisfies the equation

$$\delta(\delta+c-1)(\delta+c-2)z - x[p(\delta+b)+q(\delta+b')](\delta+c-1)z + \\ +pqx^2(\delta+b+b')z = 0 \qquad (21)$$

whose other solutions are readily deduced from the results of § 3. We may, however, note that, if $c = b+b'$, then (21) may be written

$$(\delta+b+b'-2)[\delta(\delta+b+b'-1) - x\{p(\delta+b)+q(\delta+b')\}+pqx^2]z = 0,$$

and that, as originally pointed out to me by Mr. Chaundy, the functions

$$\Phi_2(b,b';b+b';px,qx), \qquad x^{1-b-b'}\Phi_2(1-b',1-b;2-b-b';px,qx)$$

are solutions of the second-order equation

$$\delta(\delta+b+b'-1)z - x[p(\delta+b)+q(\delta+b')]z + pqx^2z = 0.$$

6. The equation of ${}_0F_1(c,px)\,{}_0F_1(c,qx)$

The function $\qquad z = {}_0F_1(c,px)\,{}_0F_1(c,qx)$

satisfies the partial equations

$$\theta(\theta+c-1)z = pxz, \qquad \phi(\phi+c-1)z = qxz. \qquad (22)$$

By addition we have

$$\delta(\delta+c-1)z - (p+q)xz = 2\theta\phi z. \qquad (23)$$

Now

$$\begin{aligned}(\delta+c-1)(\delta+c-2)2\theta\phi z &= 2\phi(\delta+c-2)\theta(\theta+\phi+c-1)z\\ &= 2\phi(\delta+c-2)(\theta\phi+px)z\\ &= 2\theta(\delta+c-2)(\theta\phi+qx)z\\ &= \delta(\delta+c-2)\theta\phi z+px\phi(\delta+c-1)z+qx\theta(\delta+c-1)z\\ &= [\delta(\delta+c-2)+(p+q)x]\theta\phi z+2pqx^2z,\end{aligned}$$

i.e.

$$[(\delta+c-2)(\delta+2c-2)-(p+q)x]2\theta\phi z = 4pqx^2z. \tag{24}$$

Combining (23) and (24) we have

$$[(\delta+2c-2)(\delta+c-2)-(p+q)x][\delta(\delta+c-1)-(p+q)x]z = 4pqx^2z,$$

i.e.

$$\begin{aligned}&\delta(\delta+c-1)(\delta+c-2)(\delta+2c-2)z-\\ &\qquad -(p+q)x(2\delta+2c-1)(\delta+c-1)z+(p-q)^2x^2z = 0,\end{aligned} \tag{25}$$

as the differential equation satisfied by ${}_0F_1(c,px)\,{}_0F_1(c,qx)$. Other independent solutions are

$$x^{1-c}\,{}_0F_1(c;px)\,{}_0F_1(2-c;qx), \qquad x^{1-c}\,{}_0F_1(2-c,px)\,{}_0F_1(c;qx),$$
$$x^{2-2c}\,{}_0F_1(2-c;px)\,{}_0F_1(2-c;qx).$$

If the equations (22) are replaced by

$$\theta(\theta+c-1)z = pxf(\theta+\phi)z, \qquad \phi(\phi+c-1)z = qxf(\theta+\phi)z,$$

we obtain in place of (25)

$$\begin{aligned}&\delta(\delta+c-1)(\delta+c-2)(\delta+2c-2)z-\\ &\qquad -(p+q)x(\delta+c-1)(2\delta+2c-1)f(\delta)z+\\ &\qquad\qquad +(p-q)^2x^2f(\delta)f(\delta+1)z = 0.\end{aligned} \tag{26}$$

In particular, taking $f = (\theta+\phi+a)(\theta+\phi+b)$, we have the equations satisfied by Appell's function

$$F^{(4)}(a;b;c,c;px,qx) = \sum_0^\infty \sum_0^\infty \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_m(c)_n}(px)^m(qx)^n,$$

which is therefore a solution of

$$\begin{aligned}&\delta(\delta+c-1)(\delta+c-2)(\delta+2c-2)z-\\ &\qquad -(p+q)x(\delta+c-1)(2\delta+2c-1)(\delta+a)(\delta+b)z+\\ &\qquad\qquad +(p-q)^2x^2(\delta+a)(\delta+a+1)(\delta+b)(\delta+b+1)z = 0.\end{aligned} \tag{27}$$

Other solutions of this equation are* immediately obtainable.

* (1), 52.

Returning to (25), we recall that

$$ {}_0F_1(c, -\tfrac{1}{4}x^2) = \Gamma(c)(\tfrac{1}{2}x)^{1-c}J_{c-1}(x), \tag{28}$$

where J is Bessel's function, and, making the necessary changes of variable and writing $c-1 = \mu$, we obtain as the differential equation satisfied by the products $J_{\pm\mu}(ax)J_{\pm\mu}(bx)$

$$\delta(\delta-2)(\delta^2-4\mu^2)z+2(a^2+b^2)x^2\delta(\delta+1)z+(a^2-b^2)^2x^4z = 0. \tag{29}$$

If in this $a = b = 1$, then it may be written

$$(\delta-2)[\delta(\delta^2-4\mu^2)z+4x^2(\delta+1)z] = 0,$$

and we recognize the inner bracket as the differential equation satisfied* by $J_\mu^2(x)$.

7. The equation of ${}_0F_1(c; px)\,{}_0F_1(c'; qx)$

The partial equations satisfied by this function are

$$\theta(\theta+c-1)z = pxz, \qquad \phi(\phi+c'-1)z = qxz. \tag{30}$$

Thus
$$(\delta+c+c'-2)\theta\phi z = \phi pxz+\theta qxz = (px\phi+qx\theta)z, \tag{31}$$

and
$$\begin{aligned} Uz &\equiv [\delta(\delta+c-1)-(p+q)x]z = (2\theta+c-c')\phi z,\\ U'z &\equiv [\delta(\delta+c'-1)-(p+q)x]z = (2\phi+c'-c)\theta z. \end{aligned}\tag{32}$$

Also
$$\begin{aligned} \theta(\delta+c-1)(\delta+c'-1)z &= (\delta+c'-1)[\theta\phi+\theta(\theta+c-1)]z\\ &= (\delta+c'-1)\theta\phi z+px(\delta+c')z,\\ \phi(\delta+c-1)(\delta+c'-1)z &= (\delta+c-1)\theta\phi z+qx(\delta+c)z. \end{aligned}$$

By addition
$$\begin{aligned} Vz &\equiv \delta(\delta+c-1)(\delta+c'-1)z-px(\delta+c')z-qx(\delta+c)z\\ &= (2\delta+c+c'-2)\theta\phi z. \end{aligned}\tag{33}$$

Thus
$$\begin{aligned} (\delta+c+c'-2)Vz &= (2\delta+c+c'-2)(px\phi+qx\theta)z, \quad \text{by (31)},\\ &= px[2\phi(\phi+c'-1)+(2\theta+c-c')\phi+2\phi]z\\ &\qquad +qx[2\theta(\theta+c-1)+(2\phi+c'-c)\theta+2\theta]z\\ &= 4pqx^2z+pxUz+qxU'z+2\theta\phi(\delta+c+c'-2)z,\\ &\qquad\qquad \text{by (31) and (32)}, \end{aligned}$$

i.e.
$$\begin{aligned} Wz &\equiv \delta(\delta+c-1)(\delta+c'-1)(\delta+c+c'-2)z-\\ &\quad -px[(\delta+c')(\delta+c+c'-1)+\delta(\delta+c-1)]z-\\ &\quad -qx[(\delta+c)(\delta+c+c'-1)+\delta(\delta+c'-1)]z+(p-q)^2x^2z\\ &= 2\theta\phi(\delta+c+c'-2)z. \end{aligned}\tag{34}$$

* (2), 146.

From (33) and (34),

$$(2\delta+c+c'-2)Wz = 2(\delta+c+c'-2)Vz,$$

a relation in which θ, ϕ appear only in the association $\theta+\phi = \delta$, and, on rearrangement, we find that

$z = {}_0F_1(c;px)\,{}_0F_1(c';qx)$ *is a solution of*

$$\begin{aligned}\delta(\delta+c-1)(\delta+c'-1)(\delta+c+c'-2)(\delta+\tfrac{1}{2}c+\tfrac{1}{2}c'-2)z-&\\ -2(p+q)x(\delta+\tfrac{1}{2}c+\tfrac{1}{2}c')(\delta+\tfrac{1}{2}c+\tfrac{1}{2}c'-\tfrac{1}{2})(\delta+\tfrac{1}{2}c+\tfrac{1}{2}c'-1)z+&\\ +\tfrac{1}{2}(p-q)x(c-c')(c+c'-2)(\delta+\tfrac{1}{2}c+\tfrac{1}{2}c'-\tfrac{1}{2})z+&\\ +(p-q)^2x^2(\delta+\tfrac{1}{2}c+\tfrac{1}{2}c'+1)z = 0.&\end{aligned} \tag{35}$$

Other solutions are evidently

$$x^{1-c}\,{}_0F_1(2-c;px)\,{}_0F_1(c';qx), \qquad x^{1-c'}\,{}_0F_1(c;px)\,{}_0F_1(2-c';qx),$$
$$x^{2-c-c'}\,{}_0F_1(2-c;px)\,{}_0F_1(2-c';qx),$$

but there is no obvious solution corresponding to the exponent at the origin $2-\frac{1}{2}c-\frac{1}{2}c'$. Reserving this point for discussion until (35) has been transformed into the form appropriate to Bessel's functions, I observe that, if for brevity we write (35) as

$$f_0(\delta)z-2(p+q)xf_1(\delta)z+\tfrac{1}{2}(p-q)xf_2(\delta)z+(p-q)^2x^2f_3(\delta)z = 0,$$

then, by a reasoning parallel to that of § 6,

$$\begin{aligned}f_0(\delta)z-2(p+q)xf_1(\delta)(\delta+a)(\delta+b)z+\tfrac{1}{2}(p-q)xf_2(\delta)(\delta+a)(\delta+b)z+&\\ +(p-q)^2x^2f_3(\delta)(\delta+a)(\delta+a+1)(\delta+b)(\delta+b+1)z = 0&\end{aligned} \tag{36}$$

is satisfied by $\qquad z = F^{(4)}(a;b;c,c';px,qx),$

and that *three* other solutions are immediately obtainable.*

If in (35) we set $c' = c$, it may be written

$$\begin{aligned}(\delta+c-1)[\delta(\delta+c-1)(\delta+c-2)(\delta+2c-2)z-&\\ -2(p+q)x(\delta+c-1)(\delta+c-\tfrac{1}{2})z+(p-q)^2x^2z] = 0,&\end{aligned}$$

where the inner bracket is recognized as (26). A similar substitution in (36) leads to (27). Again in (36) set $p = q = 1$, when it factorizes as

$$\begin{aligned}[\delta+\tfrac{1}{2}c+\tfrac{1}{2}c'-2][\delta(\delta+c-1)(\delta+c'-1)(\delta+c+c'-2)-&\\ -4x(\delta+a)(\delta+b)(\delta+\tfrac{1}{2}c+\tfrac{1}{2}c')(\delta+\tfrac{1}{2}c+\tfrac{1}{2}c'-\tfrac{1}{2})]z = 0.&\end{aligned}$$

* (1), 52.

Since the equation possesses only one solution in the form of a power series led by a constant term, we evidently have

$$F^{(4)}(a;b;c,c';x,x) = {}_4F_3\left[\begin{matrix}a, b, \tfrac{1}{2}c+\tfrac{1}{2}c', \tfrac{1}{2}c+\tfrac{1}{2}c'-\tfrac{1}{2};\ 4x\\ c, c', c+c'-1\end{matrix}\right] \tag{37}$$

of which (**3**) 124 (66) and (**2**) 147 (1) are limiting cases.

8. The equation of $J_\mu(ax)J_\rho(bx)$

If in (35) we write

$$x = x'^2,\ p = -\tfrac{1}{4}a^2,\ q = -\tfrac{1}{4}b^2,\ c-1 = \mu,\ c'-1 = \rho,$$
$$z = (x')^{-\mu-\rho}z',$$

then, recalling (28) and suppressing accents after transformation, we find that

The products of Bessel's functions $J_{\pm\mu}(ax)J_{\pm\rho}(bx)$ are four independent solutions of the differential equation

$$\begin{aligned}\Delta z \equiv{}& (\delta-2)[\delta^4-2(\mu^2+\rho^2)\delta^2+(\mu^2-\rho^2)^2]z+2(a^2+b^2)x^2\delta(\delta+1)(\delta+2)z-\\ &-2(a^2-b^2)(\mu^2-\rho^2)x^2(\delta+1)z+(a^2-b^2)^2x^4(\delta+4)z\\ ={}& 0.\end{aligned} \tag{38}$$

If in this we set $a = b = 1$, the resulting equation may be written

$$(\delta-2)[\{\delta^4-2(\mu^2+\rho^2)\delta^2+(\mu^2-\rho^2)^2\}z+4x^2(\delta+1)(\delta+2)z] = 0, \tag{39}$$

where the inner bracket furnishes the equation of the fourth order* satisfied by the product of two Bessel's functions with the same argument. Again, on setting $\rho = \mu$ in (38) it may be written

$$\delta[\delta(\delta-2)(\delta^2-4\mu^2)z+2(a^2+b^2)x^2\delta(\delta+1)z+(a^2-b^2)^2x^4z] = 0,$$

and we recognize the bracket as (29).

There remains the problem of a fifth solution of (38). We begin by comparing that equation with the equation of the fourth order whose only solutions are $J_{\pm\mu}(ax)J_{\pm\rho}(bx)$. On applying the methods of (**2**) 146 the latter is found to be

$$\begin{aligned}\Delta_1 z&\\ \equiv{}& (\rho^2-\mu^2)[\{\delta^4-2(\mu^2+\rho^2)\delta^2+(\mu^2-\rho^2)^2\}+2x^2(a^2+b^2)(\delta+1)(\delta+2)]z+\\ &+(a^2-b^2)x^2[\delta^4-2\delta^3-2(\mu^2+\rho^2)\delta^2+4(\mu^2+\rho^2)\delta+3(\mu^2-\rho^2)^2]z+\\ &+(a^2-b^2)x^4[2(a^2+b^2)\delta(\delta+1)-3(a^2-b^2)(\mu^2-\rho^2)]z+(a^2-b^2)^3x^6z\\ ={}& 0,\end{aligned} \tag{40}$$

* (**2**), 146.

a result which, in this or an equivalent form, is presumably to be found in the literature, though I have failed to discover it. As a check upon the calculations we may set in turn $\rho = \mu$ and $b = a$, when we recover (29) and (39).

It will be observed that, whereas $\Delta z = 0$ is a 'three-term' equation of type (1), the equation $\Delta_1 z = 0$, though free of the irrelevant solution, is a 'four-term' equation and thus in one respect less elementary in character. Since Δ annihilates all functions annihilated by Δ_1, the latter operator is a 'post-factor' of Δ, and inspection of the terms of highest order in the operators shows that

$$\Delta = \{(a^2-b^2)x^2-(\mu^2-\rho^2)\}^{-1}\{\delta-2-u(x)\}\Delta_1,$$

where $u(x)$ is an undetermined function of x, which, by a further comparison of coefficients, is shown to be zero. We thus have the identity of operators

$$\begin{aligned}[(a^2-b^2)x^2-(\mu^2-\rho^2)]\Delta &\\ &= (\delta-2)\Delta_1 \\ &= (\delta-2)[\{(a^2-b^2)x^2-(\mu^2-\rho^2)\}\delta^4-2(a^2-b^2)x^2\delta^3+\ldots]. \end{aligned} \tag{41}$$

Now, for brevity, set $J_{\pm\mu}(ax)J_{\pm\rho}(bx) = J_{\pm\mu}\,J_{\pm\rho}$, and let functions $u_{\pm\mu,\pm\rho}$ be determined to satisfy the equations

$$\left.\begin{aligned} &\sum J_{\pm\mu}\,J_{\pm\rho}\,\delta u_{\pm\mu,\pm\rho} = 0 \\ &\sum \delta(J_{\pm\mu}\,J_{\pm\rho})\delta u_{\pm\mu,\pm\rho} = 0 \\ &\sum \delta^2(J_{\pm\mu}\,J_{\pm\rho})\delta u_{\pm\mu,\pm\rho} = 0 \\ &[(a^2-b^2)x^2-(\mu^2-\rho^2)]x^{-2}\sum \delta^3(J_{\pm\mu}\,J_{\pm\rho})\delta u_{\pm\mu,\pm\rho} = \text{constant}. \end{aligned}\right\} \tag{42}$$

Then $z = \sum J_{\pm\mu}\,J_{\pm\rho}\,u_{\pm\mu,\pm\rho}$ will be a solution of $\Delta z = 0$.

Now, from (41), the δ-Wronskian of the four functions $J_{\pm\mu}\,J_{\pm\rho}$ is a constant multiple of $(a^2-b^2)x^2-(\mu^2-\rho^2)$, and we find without difficulty that the δ-Wronskian of the three functions $J_{\mu}\,J_{-\rho}$, $J_{-\mu}\,J_{\rho}$, $J_{-\mu}\,J_{-\rho}$ is a constant multiple of $J_{-\rho}(bx)\delta J_{-\mu}(ax)-J_{-\mu}(ax)\delta J_{-\rho}(bx)$. Thus for the solution of (42) we have, sufficiently,

$$\delta u_{\pm\mu,\pm\rho} = \pm\frac{x^2\{J_{\mp\rho}(bx)\delta J_{\mp\mu}(ax)-J_{\mp\mu}(ax)\delta J_{\mp\rho}(bx)\}}{[(a^2-b^2)x^2-(\mu^2-\rho^2)]^2}, \tag{43}$$

where the sign of the expression on the right is to be positive if the suffix on the left contains like signs, and otherwise negative.

In particular, writing $\delta_t = td/dt$, we have

$$u_{+\mu,+\rho}$$

$$= \int^x \frac{t[J_{-\rho}(bt)\delta_t J_{-\mu}(at) - J_{-\mu}(at)\delta_t J_{-\rho}(bt)]\,dt}{[(a^2-b^2)t^2-(\mu^2-\rho^2)]^2}$$

$$= (b^2-a^2)^{-1}[(a^2-b^2)x^2-(\mu^2-\rho^2)]^{-1}[J_{-\rho}(bx)\delta J_{-\mu}(ax)-J_{-\mu}(ax)\delta J_{-\rho}(bx)]$$

$$+(a^2-b^2)^{-1}\int^x \frac{[J_{-\rho}(bt)\delta_t^2 J_{-\mu}(at)-J_{-\mu}(at)\delta_t^2 J_{-\rho}(bt)]\,dt}{t\{(a^2-b^2)t^2-(\mu^2-\rho^2)\}}$$

$$= (b^2-a^2)^{-1}[(a^2-b^2)x^2-(\mu^2-\rho^2)]^{-1}[J_{-\rho}(bx)\delta J_{-\mu}(ax)-J_{-\mu}(ax)\delta J_{-\rho}(bx)]$$

$$+(b^2-a^2)^{-1}\int^x \frac{J_{-\mu}(at)J_{-\rho}(bt)\,dt}{t},$$

with similar formulae for the other u's.

On multiplying the $u_{\pm\mu,\pm\rho}$ by $J_{\pm\mu}\,J_{\pm\rho}$ and summing, the sum of the terms arising from the integrated parts of $u_{\pm\mu,\pm\rho}$ is zero and, if we suppress the unnecessary factor $(b^2-a^2)^{-1}$, we find that

$$z = \sum \pm J_{\pm\mu}(ax)J_{\pm\rho}(bx)\int^x \frac{J_{\mp\mu}(at)J_{\mp\rho}(bt)\,dt}{t},$$

i.e.
$$z = \int^x \begin{vmatrix} J_\mu(ax), & J_\mu(at) \\ J_{-\mu}(ax), & J_{-\mu}(at) \end{vmatrix} \times \begin{vmatrix} J_\rho(bx), & J_\rho(bt) \\ J_{-\rho}(bx), & J_{-\rho}(bt) \end{vmatrix} \frac{dt}{t}, \tag{44}$$

is the required fifth solution of (38). Since the lower limit of integration is arbitrary, (44) may, moreover, be regarded as the general solution. It is an interesting and not too laborious exercise to show directly, by differentiation under the sign of integration, that (44) is a solution of (38).

9. An equation of the fifth order associated with two differential equations of the second order

The results of the last paragraph suggest the following problem.

Let $D = d/dx$ and

$$(D^2-I)y = 0, \qquad (D^2-J)z = 0$$

be two linear differential equations of the second order in canonical form. Let them have respectively pairs of independent solutions $y_1(x)$, $y_2(x)$, and $z_1(x)$, $z_2(x)$ such that

$$y_2 Dy_1 - y_1 Dy_2 = z_2 Dz_1 - z_1 Dz_2 = 1,$$

and let

$$Y(x,t) = \begin{vmatrix} y_1(x), & y_1(t) \\ y_2(x), & y_2(t) \end{vmatrix}, \qquad Z(x,t) = \begin{vmatrix} z_1(x), & z_1(t) \\ z_2(x), & z_2(t) \end{vmatrix}.$$

We note that

$$\left.\begin{aligned} Y(x,x) = Z(x,x) = 0 \\ [D\,Y(x,t)]_{t=x} = [D\,Z(x,t)]_{t=x} = 1 \\ D^2Y(x,t) = I(x)Y(x,t), \qquad D^2Z(x,t) = J(x)Z(x,t) \end{aligned}\right\}. \tag{45}$$

Let now

$$W = \int^x Y(x,t)Z(x,t)\,dt. \tag{46}$$

We seek a linear differential equation satisfied by W. Since the lower limit of integration is arbitrary, any product of solutions of the two second-order equations will be a solution of the desired equation. We anticipate therefore an equation of the fifth order.

Now, bearing in mind (45), we have

$$DW = \int^x (ZDY+YDZ)\,dt, \tag{47}$$

$$(D^2-I-J)W = 2\int^x (DYDZ)\,dt,$$

$$D(D^2-I-J)W = 2+2I\int^x (YDZ)\,dt+2J\int^x (ZDY)\,dt, \tag{48}$$

$$D^2(D^2-I-J)W = 2I'\int^x (YDZ)\,dt+2J'\int^x (ZDY)\,dt+$$
$$+4IJW+(I+J)(D^2-I-J)W,$$

i.e., writing for brevity

$$\Theta = D^2-I-J,$$

$$(\Theta^2-4IJ)W = 2I'\int^x (YDZ)\,dt+2J'\int^x (ZDY)\,dt. \tag{49}$$

On eliminating the integrals from (47)–(49) we have

$$(I-J)(\Theta^2-4IJ)W-(I'-J')D\Theta W-$$
$$-2(IJ'-I'J)DW = -2(I'-J'). \tag{50}$$

Now operate with $(I'-J')D-(I''-J'')$: the right-hand side of (50) is eliminated and we have, as the required equation for W,

$$[(I'-J')D-(I''-J'')][(I-J)(\Theta^2-4IJ)-(I'-J')D\Theta-$$
$$-2(IJ'-I'J)D]W = 0. \tag{51}$$

We note that, if the second bracket stood alone, we should have* the equation satisfied by the products of solutions of the equations of the second order only.

* (2), 146.

The equation (51) may be more simply expressed as

$$(I-J)\{[(I'-J')D-(I''-J'')][\Theta^2-4IJ]W-$$
$$-(I'^2-J'^2)D^2W-2(I'J''-J'I'')DW+(I-J)(I'-J')^2W\}=0, \tag{52}$$

a form admitting considerable further simplification in an important class of cases for which $I'J''=J'I''$.

10. Whittaker's functions

As a final application of the methods of the earlier paragraphs we determine the equation satisfied by

$$z=y_1(a;c;px)\times y_1(a';c;qx),$$

where

$$y_1(a;c;px)=(px)^{\frac{1}{2}(c-1)}\exp(-\tfrac{1}{2}px)\,{}_1F_1(a;c;x). \tag{53}$$

Since $y_1(a;c;px)$ satisfies the differential equation

$$\delta^2y=(\rho^2+kx+P^2x^2)y,$$

where $k=\frac{1}{2}p(2a-c)$, $P=\frac{1}{2}p$, $\rho=\frac{1}{2}(c-1)$, we may, remembering that $\delta y=dy/dw$, where $x=e^w$, derive the differential equation satisfied by z from the general result of § 9. It is, however, somewhat less laborious to proceed as follows.

If $k'=\frac{1}{2}q(2a'-c)$, $Q=\frac{1}{2}q$, then z satisfies the partial differential equations

$$\theta^2z=(\rho^2+kx+P^2x^2)z, \qquad \phi^2z=(\rho^2+k'x+Q^2x^2)z, \tag{54}$$

whence, by addition,

$$(\delta^2-2\rho^2)z-(k+k')xz-(P^2+Q^2)x^2z=2\theta\phi z. \tag{55}$$

Thus

$$\delta(\delta^2-2\rho^2)z-(k+k')x(\delta+1)z-(P^2+Q^2)x^2(\delta+2)z$$
$$=2\phi\theta^2z+2\theta\phi^2z,$$

i.e.

$$\delta(\delta^2-4\rho^2)z-(k+k')x(\delta+1)z-(P^2+Q^2)x^2(\delta+2)z$$
$$\equiv Wz=2(kx+P^2x^2)\phi z+2(k'x+Q^2x^2)\theta z.$$

So

$$(\delta-1)(\delta-2)Wz=2x(\delta-1)\{k\delta\phi+k'\delta\theta\}z+2x^2(\delta+1)\{P^2\delta\phi+Q^2\delta\theta\}z$$
$$=2x(\delta-1)\{k\phi^2+k'\theta^2+(k+k')\theta\phi\}z+$$
$$+2x^2(\delta+1)\{P^2\phi^2+Q^2\theta^2+(P^2+Q^2)\theta\phi\}z. \tag{56}$$

If now on the right we replace $\theta^2 z$, $\phi^2 z$, $2\theta\phi z$ by their expressions taken from (54) and (55), then (56) is expressed in terms of the operator δ only, and on rearrangement we obtain as the required equation

$$\begin{aligned}\delta(\delta-1)(\delta-2)(\delta^2-4\rho^2)z-(k+k')x\,\delta(\delta-1)(2\delta+1)z-&\\ -2(P^2+Q^2)x^2\,\delta(\delta+1)^2z+(k-k')^2x^2\delta^2z+&\\ +(k-k')(P^2-Q^2)x^3(2\delta+3)z+&\\ +(P^2-Q^2)^2x^4(\delta+3)z=0.&\end{aligned} \tag{57}$$

If

$$y_2(a,c,px) = (px)^{-\frac{1}{2}(c-1)}\exp(-\tfrac{1}{2}px)\,{}_1F_1(a+1-c;2-c;px),$$

the general solution of (57) is

$$z = \int^x \begin{vmatrix} y_1(a,c,px), & y_2(a,c,px) \\ y_1(a,c,pt), & y_2(a,c,pt) \end{vmatrix} \times \begin{vmatrix} y_1(a',c,qx), & y_2(a',c,qx) \\ y_1(a',c,qt), & y_2(a',c,qt) \end{vmatrix} \frac{dt}{t}. \tag{58}$$

We may note that, employing the terminology of § 1, the equation for y is a three-term equation while that for the product z has five terms.

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INTEGRAL EQUATIONS FOR HEUN FUNCTIONS

By A. ERDÉLYI (*Edinburgh*)

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1. SOME years ago C. G. Lambe and D. R. Ward (3) obtained integral equations for Heun polynomials, i.e. for polynomial solutions of the Fuchsian differential equation of the second order with four singularities. The present note contains an extension of the results of their paper. The new results include (i) integral equations for typical polynomial or transcendental solutions of the *general* Heun equation, i.e. with arbitrary values of the accessory parameter h, (ii) more general nuclei than those considered by Lambe and Ward, and (iii) the most general nucleus for Heun polynomials. The work presented here is analogous to and arose from certain investigations on periodic Lamé functions, which will be published in the *Proceedings of the Edinburgh Mathematical Society*. Also there is a close connexion between the integral equations presented here and certain expansions of solutions of the Heun equation in series of hypergeometric functions.*

2. Let M_x be the differential operator

$$M_x \equiv x(x-1)(x-a)\frac{\partial^2}{\partial x^2}+$$
$$+\{\gamma(x-1)(x-a)+\delta x(x-a)+\epsilon x(x-1)\}\frac{\partial}{\partial x}+\alpha\beta x, \qquad (1)$$

and M_y the differential operator obtained by replacing x by y. We assume throughout

$$1+\alpha+\beta-\gamma-\delta-\epsilon = 0. \qquad (2)$$

The Heun equation is

$$(M_x-\alpha\beta h)u = 0. \qquad (3)$$

Its typical solution which is analytical in the neighbourhood of and equal to unity at the origin is usually denoted by $F(h, x)$, or, more fully, by $F(a, h; \alpha, \beta, \gamma, \delta; x)$. Apart from exceptional (logarithmic) cases, the general solution of the Heun equation can be expressed in terms of functions of the type $F(h, x)$, and therefore only integral equations for these functions will be considered.

* For the latter expansions compare A. Erdélyi, *Duke Math. Journal*, 9 (1942), 48–58.

The integral equations discussed will be of the form

$$F(h,x) = \lambda \int_C y^{\gamma-1}(1-y)^{\delta-1}(1-y/a)^{\epsilon-1}K(x,y)F(h,y)\,dy. \tag{4}$$

It is seen either from the paper of Lambe and Ward or from the more general investigations of Ince (**1**), (**2**) that the nucleus K must satisfy the partial differential equation

$$(M_x - M_y)K = 0, \tag{5}$$

and the contour of integration C must be such that the 'integrated parts' vanish. We notice that h does not appear in (5); thus K may be taken to be independent of h and to depend solely on α, β, γ, δ, x, y, a.

3. In order to find typical solutions of the partial differential equation $(M_x - M_y)K = 0$, let us introduce new variables θ, ϕ, where

$$\cos\theta = \left(\frac{xy}{a}\right)^{\frac{1}{2}}, \qquad \sin\theta\cos\phi = i\left\{\frac{(x-a)(y-a)}{a(1-a)}\right\}^{\frac{1}{2}},$$
$$\sin\theta\sin\phi = \left\{\frac{(x-1)(y-1)}{1-a}\right\}^{\frac{1}{2}}. \tag{6}$$

Here θ and ϕ may be conceived as polar coordinates on the unit sphere. Then (5) becomes

$$\sin^2\theta\left[\frac{\partial^2 K}{\partial\theta^2} + \{(1-2\gamma)\tan\theta + 2(\delta+\epsilon-\tfrac{1}{2})\cot\theta\}\frac{\partial K}{\partial\theta} - 4\alpha\beta K\right] +$$
$$+\frac{\partial^2 K}{\partial\phi^2} + \{(1-2\delta)\cot\phi - (1-2\epsilon)\tan\phi\}\frac{\partial K}{\partial\phi} = 0. \tag{7}$$

This equation can be integrated by a product of a function of θ and a function of ϕ. Typical solutions are

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & \frac{1}{2}-\delta-\sigma & \alpha & \cos^2\theta \\ 1-\gamma & \frac{1}{2}-\epsilon+\sigma & \beta & \end{matrix}\right\} \times P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & -\frac{1}{2}+\delta+\sigma & \cos^2\phi \\ 1-\epsilon, & 1-\delta & -\frac{1}{2}+\epsilon-\sigma & \end{matrix}\right\}, \tag{8}$$

where $P\{\ \}$ is the usual Riemannian symbol, σ is an arbitrary separation constant, and

$$\cos^2\theta = \frac{xy}{a}, \qquad \cos^2\phi = \frac{(x-a)(y-a)}{(1-a)(xy-a)}. \tag{9}$$

Two simple types of special nuclei will be mentioned. If $\sigma = \frac{1}{2}-\delta$, there is a branch of the second P-function which is constant; asso-

ciating it with that branch of the first P-function which is regular at the origin, the special nucleus

$$F(\alpha,\beta;\gamma;xy/a) \tag{10}$$

is obtained. A similar simplification is possible for $\sigma = \epsilon-\frac{1}{2}$. The nucleus (10) for $\alpha = 0, -1, -2,...$ is the nucleus discussed by Lambe and Ward. Again, if σ is one of the roots of the equation,

$$(\delta-\tfrac{1}{2}+\sigma-\alpha)(\delta-\tfrac{1}{2}+\sigma-\beta)(\epsilon-\tfrac{1}{2}-\sigma-\alpha)(\epsilon-\tfrac{1}{2}-\sigma-\beta) = 0,$$

one branch of the first P-function in (8) is an elementary function, and we have nuclei which, apart from an elementary factor, are hypergeometric functions of $(x-a)(y-a)/\{(1-a)(xy-a)\}$.

4. Let us return now to the integral equation. Suppose that $K(x,y)$ is a function of the form (8), or else a sum or integral with respect to σ of such functions. Then

$$\int\limits_C y^{\gamma-1}(1-y)^{\delta-1}(1-y/a)^{\epsilon-1}K(x,y)F(h,y)\,dy \tag{11}$$

s a solution of the Heun equation provided that the 'integrated parts' vanish. This latter condition is satisfied, e.g., if C is a contour closed on the Riemannian surface of the integrand. Let us assume furthermore that the singularity $y = a/x$ of the integrand is *outside* C, and that the branch regular at the origin of the first P-function in (8) has been chosen. Then (11) will represent an analytic function of x regular at the origin, where it takes the value

$$\int\limits_C y^{\gamma-1}(1-y)^{\delta-1}(1-y/a)^{\epsilon-1}K(0,y)F(h,y)\,dy. \tag{12}$$

Therefore (11) is a multiple of $F(h,x)$, and (4) is true, $1/\lambda$ being given by (12). It is important to remark that $K(x,y)$ and C can always be chosen so that (12) does not vanish. In fact, a branch of the second P-function in (8) can be chosen so that

$$(1-y)^{\delta-1}K(0,y)F(h,y)$$

is analytic at $y = 0$ but not analytic at $y = 1$; with such a nucleus the Pochhammer double-loop slung round $y = 0$ and $y = 1$ is a suitable contour unless γ or δ or both are positive integers: in the latter cases a simple loop or the interval $(0,1)$ will be a suitable choice for C.

Thus $F(h, x)$, *with arbitrary values of* h, satisfies the integral equation (4) provided that K is either a nucleus of the form

$$(xy-a)^{\frac{1}{2}-\delta-\sigma}F\left(\tfrac{1}{2}-\delta-\sigma+\alpha, \tfrac{1}{2}-\delta-\sigma+\beta; \gamma; \frac{xy}{a}\right)\times$$

$$\times P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & -\frac{1}{2}+\delta+\sigma & \dfrac{(x-a)(y-a)}{(1-a)(xy-a)} \\ 1-\epsilon & 1-\delta & -\frac{1}{2}+\epsilon-\sigma & \end{Bmatrix}, \quad (13)$$

or else a combination (sum or integral with respect to σ) of such nuclei, and provided that C is a contour closed on the Riemann surface of the integrand, such that $y = a/x$ lies outside it and that (12) does not vanish.

5. At first it may seem strange that we have obtained integral equations for $F(h, x)$ without subjecting this function to boundary conditions and thereby selecting certain characteristic values of the accessory parameter h. The explanation is that (4) is, in general, a *singular* integral equation. Adjoining appropriate boundary conditions, it is possible to obtain from (4) integral equations of the usual Fredholm type.

A typical pair of boundary conditions is *regularity at* $x = 0$ *and at* $x = 1$. Solutions of (3) satisfying these boundary conditions will be called *Heun functions*, this name being analogous to that of (periodic) Lamé or Mathieu functions. Now, $F(h, x)$ is already regular at $x = 0$. The condition that it should also be regular at $x = 1$ yields a transcendental equation for h. In the present section it will be assumed that h is one of the denumerable sequence of roots of this transcendental equation. It will also be assumed that in (13) the branch belonging to the exponent 0 at the singularity 1 of the P-function has been chosen. Under these assumptions the integrand of (11) has only multiplicative branch-points at $y = 0$ and $y = 1$. We choose C as a double-loop slung round 0 and 1, and, under the additional hypothesis that $\Re(\gamma) > 0$, $\Re(\delta) > 0$, the contour integral will be a constant multiple of the integral extended from 0 to 1.

Thus it is seen that, if (i) h satisfies the above-mentioned transcendental equation, (ii) $\Re(\gamma) > 0$, $\Re(\delta) > 0$, and (iii) K is a nucleus of the form

$$(xy-a)^{\frac{1}{2}-\delta-\sigma}F(\tfrac{1}{2}-\delta-\sigma+\alpha, \tfrac{1}{2}-\delta-\sigma+\beta; \gamma; xy/a)\times$$

$$\times F\left(-\tfrac{1}{2}+\delta+\sigma, -\tfrac{1}{2}+\epsilon-\sigma; \delta; \frac{a(x-1)(y-1)}{(a-1)(xy-a)}\right), \quad (14)$$

or else a combination of such nuclei, then $F(h, x)$ satisfies a Fredholm integral equation of the form (4) with 0 and 1 as limits.

It is to be noted that no restriction has been imposed on α or β. Consequently, the solutions (eigen-functions) of the Fredholm integral equation are, in general, *transcendental* Heun *functions*, and not Heun polynomials.

6. In this last section Heun *polynomials* will be discussed. Such polynomials arise if either α or β is an integer $\leqslant 0$. To fix ideas we shall assume throughout the rest of this note $\alpha = -n$ where n is a non-negative integer. In this as in the general case all results of the previous section hold true without restriction or modification; the *additional* features will be discussed in this section.

In this as in the general case there is a transcendental equation for h as a condition for the existence of Heun functions. The special feature in the case $\alpha = -n$ is that $n+1$ of the Heun functions are *polynomials* of degree n in x.* These polynomials are called Heun polynomials. Besides these polynomials—a fact very often overlooked—there is an infinity of transcendental Heun functions even for $\alpha = -n$.

Heun polynomials are naturally fully included in the general statement of the preceding section, and hence they satisfy integral equations with nuclei of the type (14). In general, i.e. with arbitrary values of σ, these integral equations will be satisfied by transcendental Heun functions as well as by polynomial ones. The question arises, and will be fully answered, whether there are nuclei *all* eigenfunctions of which are Heun polynomials.

From the theorem on the bilinear development of the symmetric nucleus of an integral equation it follows that, if there are such nuclei, they must be polynomials of degree n in x; and conversely, if K is a polynomial of degree n in x, all solutions of (4) share the same property. There are exactly $n+1$ such polynomial nuclei of form (14), corresponding to the $n+1$ values

$$\sigma_m = \tfrac{1}{2}-\delta-m \quad (m = 0, 1,..., n) \tag{15}$$

of σ; σ_0 corresponds to the nucleus discussed by Lambe and Ward, the other n nuclei are new. At the same time (15) exhausts all possibilities, or, more precisely, any nucleus *all* characteristic func-

* Compare, for instance, (3) § 2.1.

tions of which are Heun polynomials is a linear combination, with constant coefficients, of the nuclei characterized by (15). The proof of this is immediate.

This last theorem has interesting consequences. The product $F_n(h,x)F_n(h,y)$ of two Heun polynomials can be expressed as a linear combination of the $n+1$ special nuclei (15). Putting $y=0$, 1, or a in this expression, three different expansions of a Heun polynomial in terms of Jacobi polynomials ensue.

A great variety of related theorems may be obtained from the results of the present note by appropriate transformations of the variables in the Heun equation.

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ON FUNCTIONS REPRESENTED BY CERTAIN SERIES

By T. VIJAYARAGHAVAN (*Dacca*)

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THEOREM. *If $a_n \geqslant 0$, $a_n = O(1)$, $a_n \neq 0$ for an infinity of values of n, and*

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n}, \qquad g(z) = \sum_{n=1}^{\infty} \frac{a_n z^n}{1+z^n},$$

then the unit circle is the natural boundary of $f(z)$ and also of $g(z)$.

This result is a sharpening of a theorem proved by G. Pólya.* That neither of the conditions $a_n \geqslant 0$, $a_n = O(1)$ is entirely superfluous can be seen from the following known examples:

$$\sum \frac{\phi(n)z^n}{1-z^n} = \frac{z}{(1-z)^2}, \qquad \sum \frac{\phi(n)z^n}{1+z^n} = \frac{z}{(1-z)^2} - \frac{2z^2}{(1-z^2)^2},$$

$$\sum \frac{\mu(n)z^n}{1-z^n} = z, \qquad \sum \frac{\mu(n)z^n}{1+z^n} = z - 2z^2.$$

It may be remarked here that $f(z)$ and $g(z)$ are connected by the equation $g(z) = f(z) - 2f(z^2)$, so that $g(z)$ is a rational function of z whenever $f(z)$ is a rational function of z. The converse is not true, for, if $a_n = n$ when $n = 2^m$ $(m = 0, 1, 2,...)$ and $a_n = 0$ otherwise, then it is easy to verify that $|f(re^{2k\pi i/2^m})| \to \infty$ as $r \to 1-0$, k and m being any pair of integers; hence the unit circle is the natural boundary of $f(z)$, whereas the corresponding $g(z)$ is equal to $z/(1-z)$.

In proving the theorem of this note I concern myself with $g(z)$ only, as the proof in the case of $f(z)$ is similar and slightly easier too. It is easy to see that the series for $g(z)$ is convergent for $|z| < 1$, and that its sum represents an analytic function of z. It is shown below that there exist arbitrarily large integers m such that every primitive $(2m)$th root of unity is a singular point of $g(z)$. Actually it is proved that, if ξ is such a primitive root, then

$$\lim_{r \to 1-0} |g(\xi r)| = \infty. \tag{1}$$

From the prime number theorem (or from less deep results concerning the distribution of primes) we can deduce easily that, if ϵ be any positive number and m any suitably large integer, then the primitive $(2m)$th roots of unity are so distributed on the unit circle

* *Proc. London Math. Soc.* (2) 21 (1922), 36–8.

that there is at least one such root in every arc of length ϵ. From this observation and (1) it will follow that the singular points of $g(z)$ are dense everywhere on the unit circle, and hence that the unit circle is the natural boundary of $g(z)$. The rest of this note is concerned with proving (1) for suitable values of ξ.

Let $g_m(\xi)$ denote the sum of those terms $a_n z^n/(1+z^n)$ for which n is an odd multiple of m, and $h_m(z)$ the sum of the remaining terms of the series for $g(z)$. If ξ is a primitive $(2m)$th root of unity, $0 \leqslant r < 1$, and n is not an odd multiple of m, then, by considering the position of the point $(\xi r)^n$ in the complex plane it is easy to verify that

$$|1+(\xi r)^n| > c = c(m) = 1-\cos\frac{\pi}{m},$$

and that therefore

$$\left|\frac{1}{1+(\xi r)^n}\right| < \frac{1}{c}.$$

If M is the upper bound of $a_1, a_2, a_3,\ldots$, and $0 \leqslant r < 1$, then we have

$$|(1-r)h_m(\xi r)| < (1-r)\frac{M}{c}\sum_{n=1}^{\infty} r^n < \frac{M}{c}. \tag{2}$$

To estimate $g_m(\xi r)$ we observe that, if n is an odd multiple of m, say $n = m(2l+1)$, then

$$(r-1)\frac{a_n(\xi r)^n}{1+(\xi r)^n} = \frac{(1-r)a_n r^n}{1-r^n} \geqslant \frac{a_n r^n}{n} = \frac{a_{m(2l+1)} r^{m(2l+1)}}{m(2l+1)}.$$

If now we suppose that

$$\sum_{l=0}^{\infty} \frac{a_{m(2l+1)}}{m(2l+1)} = \infty, \tag{3}$$

then, since $a_n \geqslant 0$, it follows that

$$\lim_{r\to 1-0} (r-1)g_m(\xi r) = \infty. \tag{4}$$

If (3) is true for an infinity of values of m, then (4) is true for those values of m, and then it follows from (2) that

$$\lim_{r\to 1-0} |g_m(\xi r)| = \lim_{r\to 1-0} |g_m(\xi r)+h_m(\xi r)| = \infty$$

for a set of values of ξ which is dense everywhere on the unit circle. Now in order to complete the proof of the theorem it remains only to dispose of the case in which the expression on the left-hand side of (3) is convergent for all large values of m, i.e.

$$\sum_{l=0}^{\infty} \frac{a_{m(2l+1)}}{m(2l+1)} < \infty \quad (m = H+1,\, H+2,\, H+3,\ldots). \tag{5}$$

It is proved below that in this case

$$(1-r)h_m(\xi r) \to 0 \tag{6}$$

as $r \to 1-0$, m being any positive integer, and ξ being, as before, any primitive $(2m)$th root of unity. Moreover,

$$\lim_{r\to 1-0}(r-1)g_m(\xi r) \geqslant \frac{a_m}{m}, \tag{7}$$

and, since $a_m > 0$ for an infinity of values of m, the theorem follows from (6) and (7); however, we have yet to prove (6), and for this purpose the following lemmas are needed.

LEMMA 1. *If*

$$\delta_m = \sum_{l=0}^{\infty} \frac{a_{m(2l+1)}}{m(2l+1)} < \infty, \qquad \psi_m(r) = \sum_{l=0}^{\infty} a_{m(2l+1)} r^{m(2l+1)},$$

then $(1-r)\psi_m(r) \to 0$ *as* $r \to 1-0$.

For,
$$\int_0^1 \psi_m(r)\, dr = \sum_{l=0}^{\infty} \frac{a_{m(2l+1)}}{1+m(2l+1)} \leqslant \delta_m < \infty,$$

and, since $\psi_m(r)$ is a monotonic function of r, the conclusion of the lemma follows from the convergence of the integral.

We define the density of a sequence $n_1, n_2, n_3,\ldots$ to be the limit of m/n_m, provided that the limit exists as $m \to \infty$, and denote it by $d(n_1, n_2, n_3,\ldots)$.

LEMMA 2. *If α is any positive integer, and $p_1, p_2,\ldots, p_k$ are odd numbers which are prime to each other in pairs, and $0 < n_1 < n_2 < n_3 < \ldots$ is precisely the set of integers that are such that none of them is an odd multiple of any number of the form $2^u p_v$ $(0 \leqslant u < \alpha;\ 1 \leqslant v \leqslant k)$ then*

$$d(n_1, n_2, n_3,\ldots) = 1-\Big(1-\frac{1}{2^\alpha}\Big)\Big\{1-\Big(1-\frac{1}{p_1}\Big)\Big(1-\frac{1}{p_2}\Big)\ldots\Big(1-\frac{1}{p_k}\Big)\Big\}. \tag{8}$$

Let s_t denote the sequence of integers

$$2^{t-1}p_v(2l+1) \quad (v = 1, 2,\ldots, k;\ l = 0, 1, 2,\ldots)$$

arranged in order of magnitude and without repetition. The sequences $s_1, s_2,\ldots, s_\alpha$ are all distinct (i.e. no two sequences have a common member) and, if d_u denotes the density of s_u, then it is easy to see that

$$d_u = \frac{1}{2^u}\Big\{1-\Big(1-\frac{1}{p_1}\Big)\Big(1-\frac{1}{p_2}\Big)\ldots\Big(1-\frac{1}{p_k}\Big)\Big\}.$$

If we remove from the sequence of natural numbers the sequences $s_1, s_2, \ldots, s_\alpha$, then the resulting sequence is $n_1, n_2, n_3, \ldots$ and its density is equal to $1-d_1-d_2-\ldots-d_\alpha$, which is plainly equal to the expression on the right-hand side of (8).

LEMMA 3. *If ϵ and H are given positive numbers, then there exist primes $p_1, p_2, \ldots, p_k$ and an integer α such that $H < p_1 < p_2 < \ldots < p_k$, and $d(n_1, n_2, n_3, \ldots) < \epsilon$, where the sequence $n_1, n_2, n_3, \ldots$ is defined as in the previous lemma.*

Let p_1 be a prime greater than H and $p_2, p_3, \ldots$ be the succeeding primes. Since

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right) \to 0$$

as $k \to \infty$, it is seen from Lemma 2 that $d(n_1, n_2, n_3, \ldots) < \epsilon$ whenever k and α are both suitably large.

LEMMA 4. *If $a_n \leqslant M$, $\delta = d(n_1, n_2, n_3, \ldots) < \epsilon$, and ξ is a primitive $(2m)$th root of unity, then*

$$\overline{\lim_{r\to 1-0}}\,(1-r)\sum a_{n_t} r^{n_t} < M\epsilon.$$

If s_ν denotes $\sum\limits_{n_t \leqslant \nu} a_{n_t}$, then it is plain that $\overline{\lim}\, s_\nu/\nu < M\epsilon$, and hence that

$$(1-r)\sum a_{n_t} r^{n_t} = (1-r)^2 \sum_{\nu=1}^{\infty} s_\nu r^\nu < M\epsilon$$

whenever r is sufficiently near to but less than unity.

LEMMA 5. $(1-r)h_m(r\xi) \to 0$ *as* $r \to 1-0$.

Let η be any positive number. Determine $p_1, p_2, \ldots, p_k$ and α as in Lemma 3, taking H to be the same as in (5) and $\epsilon = \eta c/M$, where $c = 1-\cos(\pi/m)$ and M is the upper bound of $a_1, a_2, a_3, \ldots$. Since $a_n \geqslant 0$, we have that $(1-r)\sum a_n r^n \leqslant \sum_1 + \sum_2$, where

$$\textstyle\sum_1 = (1-r)\sum\limits_{\substack{0\leqslant u<\alpha \\ 1\leqslant v\leqslant k}} \psi_{2^u p_v}(r), \qquad \sum_2 = (1-r)\sum\limits_{t=1}^{\infty} a_{n_t} r^{n_t}.$$

From Lemma 1 we have that $\sum_1 \to 0$ as $r \to 1-0$, and from Lemmas 3 and 4 we see that $\sum_2 < M\epsilon = \eta c$, as $r \to 1-0$. Hence, as $r \to 1-0$,

$$|(1-r)h_m(\xi r)| \leqslant \sum \frac{a_n}{c} r^n \leqslant \frac{1}{c}\left(\textstyle\sum_1+\sum_2\right) < \frac{\eta c}{c} = \eta.$$

Since η is arbitrary, the lemma is proved and this completes the proof of the theorem also.

CONTRIBUTIONS TO THE THEORY OF CLOSED HERMITIAN TRANSFORMATIONS OF DEFICIENCY-INDEX (m, m)

By H. L. HAMBURGER (*Southampton*)

[Received 6 March 1942]

1. This note contains a short statement of some new results concerning closed Hermitian transformations of deficiency-index (m, m)† and their resolvents in Hilbert space $\mathfrak{H}$. The most important of them seem to be those which are formulated in Theorems 5–7. They are derived from some new analytical representations stated in Theorems 1–4. A more detailed development of the proofs will appear in a later paper, in which the case $m = \infty$, omitted from the present note, will also be discussed. Detailed proofs of Theorems 6 and 7 are, however, given here because those in the later paper are of a different character. The abbreviations 'c.H.t.' and 'd.i.' will be used respectively for 'closed Hermitian transformations' and 'deficiency-index'.

I

2. Theorem 1. *Let H be a c.H.t. of d.i. (m, m), H^* its adjoint transformation, $\overset{0}{H}$ a self-adjoint extension of H, and $R_x = \overset{0}{H}{}_x^{-1}$, where $\overset{0}{H}_x = \overset{0}{H} - xI$, the resolvent of $\overset{0}{H}$. Let $\phi_1(x)$, $\phi_2(x)$,..., $\phi_m(x)$ be a set of m linearly independent characteristic solutions of H^* belonging to the characteristic value x, so that $H_x^* \phi_\mu(x) = 0$ ($\mu = 1, 2,..., m$) for every non-real x.*

Then the resolvent R_x and the set $\phi_\mu(x)$ determine, for every non-real x, a matrix $C(x) = \{c_{\mu\nu}(x)\}$ of order m, which has an inverse matrix $C^{-1}(x) = \{c_{\mu\nu}^{-1}(x)\}$, so that

$$(x-y)R_y \phi_\mu(x) = \phi_\mu(x) - \sum_{\nu=1}^{m} \sum_{\kappa=1}^{m} c_{\mu\nu}(x) c_{\nu\kappa}^{-1}(y) \phi_\kappa(y) \qquad (1)$$

for every non-real x and y.

To prove this we first observe that there is a matrix $A(x, y)$, whose elements depend on both variables x and y, such that

$$(x-y)R_y \phi_\mu(x) = \phi_\mu(x) - \sum_{\nu=1}^{m} a_{\mu\nu}(x, y) \phi_\nu(y). \qquad (2)$$

† The notation is that of Stone's book (**2**). See (**1**) 87, definition 15; (**2**) 81, definition 2.21, and 338, definition 9.1.

We now apply the transformation R_z to (2) and use the familiar fundamental equation for resolvents, viz.

$$(z-y)R_z R_y = R_z - R_y.$$

We thus obtain the matrix-equation

$$A(x,y)A(y,z) = A(x,z). \tag{3}$$

If we substitute $z = i$ in (3) and write $C(x)$ for $A(x,i)$, (3) becomes $A(x,y) = C(x)C^{-1}(y)$, which is the desired result.

3. If we put
$$\Phi_\mu(x) = \sum_{\nu=1}^{m} c_{\mu\nu}^{-1}(x)\phi_\nu(x), \tag{4}$$
formula (1) becomes

$$(x-y)R_y \Phi_\mu(x) = \Phi_\mu(x) - \Phi_\mu(y). \tag{5}$$

From this we immediately obtain

THEOREM 2. *The function $\{\Phi_\mu(x), g\}$, defined by (4) for any element g of $\mathfrak{H}$, is regular and analytic in every simply-connected domain of x which does not contain any point of the real axis, and*

$$d\Phi_\mu(x)/dx = R_x \Phi_\mu(x)$$

is an element of $\mathfrak{H}$ for every non-real x.

4. If $E(\lambda)$ denotes the resolution of the identity belonging to $\overset{0}{H}$, and ΔE the projection $E(b) - E(a)$, where $b > a$, we obtain

$$\overset{0}{H}_x \Delta E \Phi_\mu(x) - \overset{0}{H}_y \Delta E \Phi_\mu(y) = 0$$

by applying the transformation $\overset{0}{H}_y \Delta E$ to (5).

This formula leads to

THEOREM 3. *If we write*

$$\psi_\mu^\Delta = \overset{0}{H}_x \Delta E \Phi_\mu(x) = \int_a^b (\lambda - x) d\{E(\lambda)\Phi_\mu(x)\}, \tag{6}$$

then ψ_μ^Δ is an element of $\mathfrak{H}$ which is independent of x.

If $\mathfrak{D}$ denotes the domain of H, $\mathfrak{M}_\Delta$ the linear manifold into which the projection ΔE takes $\mathfrak{H}$, then the ψ_μ^Δ ($\mu = 1, 2, \ldots, m$) are elements of $\mathfrak{M}_\Delta$, and $\mathfrak{D}.\mathfrak{M}_\Delta$ is the set of all elements f of $\mathfrak{M}_\Delta$ which are orthogonal to each of the ψ_μ^Δ.

The last assertion in the theorem follows from the relation

$$0 = \{H_x \Delta E f, \Phi_\mu(x)\} = \{\Delta E \overset{0}{H}_x f, \Phi_\mu(x)\} = \{f, \overset{0}{H}_x \Delta E \Phi_\mu(x)\} = (f, \psi_\mu^\Delta),$$

which holds for every element $f = \Delta E f$ of $\mathfrak{D}.\mathfrak{M}_\Delta$.

5. THEOREM 4.

$$\lim_{\substack{a\to-\infty\\ b\to+\infty}} \left|\sum_{\mu=1}^{m} u_\mu \psi_\mu^\Delta\right|^2 = \int_{-\infty}^{+\infty} |\lambda-x|^2\, d\Big(E(\lambda)\sum_{\mu=1}^{m} u_\mu \Phi_\mu(x),\ \sum_{\mu=1}^{m} u_\mu \Phi_\mu(x)\Big) = \infty$$

for every system of m real or complex values of u_μ.

For suppose that this limit is finite for some system of numbers u_μ. Then $\Phi(x) = \sum_{\mu=1}^{m} u_\mu \Phi_\mu(x)$ belongs to the domain of $\overset{0}{H}$, and so $\overset{0}{H}_x \Phi(x) = 0$ for every non-real x. This is a contradiction, since the self-adjoint transformation $\overset{0}{H}$ cannot have any non-real characteristic value.

6. We now divide the whole real axis into an infinity of intervals Δ_p $(-\infty < p < +\infty)$ of finite length. Since $\Phi_\mu(x) = \sum_{p=-\infty}^{+\infty} \Delta_p E\Phi_\mu(x)$, we obtain from (6) an analytic representation of $\Phi_\mu(x)$, namely

$$\Phi_\mu(x) = \sum_{p=-\infty}^{+\infty} \Delta_p E R_x \psi_\mu^{\Delta_p} = \sum_{p=-\infty}^{+\infty} \int_{\Delta_p} \frac{d\{E(\lambda)\psi_\mu^{\Delta_p}\}}{\lambda - x}.$$

This formula may also be written as

$$\Phi_\mu(x) = \lim_{\substack{a\to-\infty\\ b\to+\infty}} \int_a^b \frac{d\{E(\lambda)\psi_\mu^{\Delta}\}}{\lambda - x}. \tag{7}$$

II

7. Before passing to our three main theorems we need a new definition: a c.H.t. H of d.i. (m, m) will be called *a prime transformation* if $\mathfrak{H}$ does not contain any closed linear manifold which reduces H and with respect to which H is self-adjoint.

In what follows we shall be concerned with prime transformations only. We can, however, easily reduce every c.H.t. H of d.i. (m, m) which is not prime to a prime transformation. Let $\mathfrak{M}$ be the greatest closed linear manifold contained in $\mathfrak{H}$ which reduces H and in which H is self-adjoint. Then H will be a prime transformation with respect to the Hilbert space $\mathfrak{H}' = \mathfrak{H} \ominus \mathfrak{M}$.

Further, it is easy to show that, if H is a c.H. prime transformation of d.i. (m, m), and $E(\lambda)$ a resolution of the identity belonging to any self-adjoint extension of H, then the equation $\Delta E\Phi_\mu(x) = 0$, holding

for every $\mu = 1, 2, \ldots, m$, for any non-real x, and for any interval Δ, implies $\Delta E g = 0$ for every element g of $\mathfrak{H}$.

8. THEOREM 5. *In order that the spectrum of any self-adjoint extension of a c.H. prime transformation H of d.i. (m, m) should contain a finite number of points only in every interval $a+\delta \leqslant \lambda \leqslant b-\delta$, it is necessary and sufficient that the m elements $\phi_\mu(\lambda+i\eta)$ should be continuous in the open rectangle $a < \lambda < b$, $|\eta| < \epsilon$.*

The fact that the condition is necessary follows from a theorem of Calkin† or from our formula (7).

In order to prove that the condition is also sufficient, we first show that the continuity of the m elements $\phi_\mu(\lambda+i\eta)$ implies that of the m^2 functions $c_{\mu\nu}(\lambda+i\eta)$ in the same open rectangle. We then deduce, by means of Cauchy's integral-representation, that, in the sense of Theorem 2, the m elements $\Phi_\mu(x)$ are also analytic in the real points $x = \lambda$ of the open interval $a < \lambda < b$, except for a set of points λ_κ in which the determinant $\|C(\lambda_\kappa)\|$ vanishes, and which is at most denumerable. These points λ_κ are poles of the $\Phi_\mu(x)$, and are all the characteristic values of $\overset{0}{H}$ in the interval $a < \lambda < b$. We further notice that

$$\{\Delta E \Phi_\mu(x), \psi_\mu^\Delta\} = \int_a^b \frac{d\{E(\lambda)\psi_\mu^\Delta, \psi_\mu^\Delta\}}{\lambda - x} \quad (\mu = 1, 2, \ldots, m),$$

is a regular analytic function for $a < x < b$, $x \neq \lambda_\kappa$. This implies, however, that $\{E(\lambda)\psi_\mu^\Delta, \psi_\mu^\Delta\} = |E(\lambda)\psi_\mu^\Delta|^2$ remains constant in every interval Δ' which does not contain any one of the roots $\lambda = \lambda_\kappa$ of $\|C(\lambda)\|$. Hence $\Delta' E \Phi_\mu(x) = \Delta' E \psi_\mu^\Delta = 0$ for every x, and so we obtain finally $\Delta' E = 0$, since H is supposed to be a prime transformation. Thus we have proved that the open interval $a < \lambda < b$ does not contain any part of the continuous spectrum of $\overset{0}{H}$.

We prove finally that the number of characteristic values λ_κ of $\overset{0}{H}$ in every interval $a+\delta \leqslant \lambda \leqslant b-\delta$ is finite by considering the two mth-order determinants

$$\Phi(x, \bar{x}) = \|\{\Phi_\mu(x), \Phi_\nu(\bar{x})\}\|, \qquad \phi(x, \bar{x}) = \|\{\phi_\mu(x), \phi_\nu(\bar{x})\}\|.$$

We readily obtain the equation

$$\frac{1}{\Phi(x, \bar{x})} = \frac{\|C(x)\| . \|\overline{C(\bar{x})}\|}{\phi(x, \bar{x})}, \tag{8}$$

† (3) 506, Theorem 3.

and then show that, within our open rectangle, $1/\phi(x,\bar{x})$ is a continuous function of $x = \lambda + i\eta$ and $1/\Phi(x,\bar{x})$ is an analytic function of x. Now it follows from (8) that, by a familiar elementary theorem of the theory of functions, $\|C(x)\|$ has only a finite number of roots in every interval $a+\delta \leqslant \lambda \leqslant b-\delta$: and this is the desired result.

Theorem 5, of course, also yields the necessary and sufficient conditions that the spectrum of every self-adjoint extension of a c.H. prime transformation of d.i. (m,m) consists of its characteristic values only, which, moreover, have no finite limit point.

III

9. The existence of c.H. prime transformations of d.i. (m,m) whose self-adjoint extensions have spectra consisting of every point of the real axis becomes evident from the following two theorems.

THEOREM 6. *Let $\overset{0}{H}$ be a self-adjoint H.t. with simple spectrum, $E(\lambda)$ its resolution of the identity, and R_x its resolvent. Also let $\overset{0}{g}$ be an element of the domain of $\overset{0}{H}$ such that the linear manifold $\mathfrak{M}(\overset{0}{g})$ spanned by $E(\lambda)\overset{0}{g}$, if λ runs from $-\infty$ to $+\infty$, coincides*† *with the whole Hilbert space $\mathfrak{H}$. Further, let $\rho(\lambda) = \{E(\lambda)\overset{0}{g}, \overset{0}{g}\}$. Then, in order to construct all the c.H. prime transformations H of d.i. $(1,1)$ to which $\overset{0}{H}$ belongs as a self-adjoint extension,*‡ *we must consider all those functions $\Omega(\lambda)$ $(-\infty < \lambda < +\infty)$ which fulfil the following three conditions:*

(i) *the set of all the points λ for which $\Omega(\lambda) = 0$ is of ρ-measure zero;*

(ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{|\Omega(\lambda)|^2}{\lambda^2+1}\, d\rho(\lambda)$ *converges;* (iii) $\displaystyle\int_{-\infty}^{+\infty} |\Omega(\lambda)|^2\, d\rho(\lambda)$ *diverges.*

Further, if we put

$$\Phi(x) = \int_{-\infty}^{+\infty} \frac{\Omega(\lambda)}{\lambda - x}\, d\{E(\lambda)\overset{0}{g}\}, \tag{9}$$

† See (**2**) 243, Theorem 7.2, and 257, Theorem 7.9.

‡ Von Neumann ((**1**) 108, Theorem 49) stated that every not-bounded H.t. T can be interpreted as the extension of another H.t. S. But he gave no method for the construction of all the transformations S having T as their extension, nor did he investigate the value of the deficiency-index of the transformation S constructed by him, which possibly might be infinite.

where x is any non-real number, we obtain the domain $\mathfrak{D}$ which determines the desired transformation H of d.i. (1, 1) *by applying the transformation R_i to the linear manifold of all the elements of $\mathfrak{H}$ which are orthogonal to $\Phi(-i)$. We have finally*

$$H_x^* \Phi(x) = 0 \tag{10}$$

for every non-real x.

To show that these three conditions are necessary, we observe that formula (9) and condition (ii) follow, after a slight alteration, from the representation (7) for $\Phi(x)$ by making use of the operational calculus; condition (iii) follows from Theorem 4, and condition (i) from the last paragraph of § 7.

10. In order to prove that these three conditions are also sufficient, we have only to show that the domain $\mathfrak{D}$ constructed in the way indicated in Theorem 6 is everywhere dense when these conditions are satisfied.

If f is any element of $\mathfrak{H}$ the operational calculus leads to the following representation of f by a function $F(\lambda)$:

$$f = \int_{-\infty}^{+\infty} F(\lambda)\, d\{E(\lambda)\overset{0}{g}\}, \tag{11}$$

$$|f|^2 = \int_{-\infty}^{+\infty} |F(\lambda)|^2\, d\rho(\lambda). \tag{12}$$

On the other hand, any function $F(\lambda)$ whose integral (12) exists furnishes an element of $\mathfrak{H}$ by (11). Now we obtain the domain $\overset{0}{\mathfrak{D}}$ of $\overset{0}{H}$ as the set of all elements

$$f' = R_i f = \int_{-\infty}^{+\infty} \frac{F(\lambda)}{\lambda - i}\, d\{E(\lambda)\overset{0}{g}\} \tag{13}$$

and the desired domain $\mathfrak{D}$ as the set of all those elements

$$h' = R_i h = \int_{-\infty}^{+\infty} \frac{G(\lambda)}{\lambda - i}\, d\{E(\lambda)\overset{0}{g}\} \tag{14}$$

which satisfy the condition

$$\{h, \Phi(-i)\} = \int_{-\infty}^{+\infty} G(\lambda) \frac{\overline{\Omega(\lambda)}}{\lambda - i}\, d\rho(\lambda) = 0. \tag{15}$$

It is sufficient to show that the domain $\mathfrak{D}$ is everywhere dense in $\overset{0}{\mathfrak{D}}$, since $\overset{0}{\mathfrak{D}}$ is itself everywhere dense in $\mathfrak{H}$. If we write

$$f = h + \frac{\{f, \Phi(-i)\}}{|\Phi(-i)|^2}\Phi(-i),$$

where f is any element of $\mathfrak{H}$ and $\{h, \Phi(-i)\} = 0$, we obtain for every element f' of $\overset{0}{\mathfrak{D}}$ the representation

$$\begin{aligned} f' &= R_i h + \frac{\{f, \Phi(-i)\}}{|\Phi(-i)|^2} R_i \Phi(-i) \\ &= \int_{-\infty}^{+\infty} \frac{G(\lambda)}{\lambda - i}\, d\{E(\lambda)\overset{0}{g}\} + \frac{\{f, \Phi(-i)\}}{|\Phi(-i)|^2} \int_{-\infty}^{+\infty} \frac{\Omega(\lambda)}{\lambda^2+1}\, d\{E(\lambda)\overset{0}{g}\}. \end{aligned} \tag{16}$$

Hence the only thing still requiring proof is that the element

$$R_i \Phi(-i) = \int_{-\infty}^{+\infty} \frac{\Omega(\lambda)}{\lambda^2+1}\, d\{E(\lambda)\overset{0}{g}\} \tag{17}$$

can be approximated to by an element (14) of $\mathfrak{D}$, since the other term of the expression (16) is already an element of $\mathfrak{D}$ itself

We first construct a complete ortho-normal set of functions $\Omega_1(\lambda)$, $\Omega_2(\lambda)$,... such that

$$\int_{-\infty}^{+\infty} \frac{\Omega_p(\lambda)\overline{\Omega_q(\lambda)}}{\lambda^2+1}\, d\rho(\lambda) = \delta_{pq} \tag{18}$$

and $$\Omega_1(\lambda) = \Omega(\lambda)\{|\Phi(i)|^2\}^{-\frac{1}{2}}.$$

If, further, we put

$$G_n(\lambda) = \frac{-\sum_{p=2}^{n} x_p \Omega_p(\lambda)}{\lambda + i}, \qquad h'_n = \int_{-\infty}^{+\infty} \frac{G_n(\lambda)}{\lambda - i}\, d\{E(\lambda)\overset{0}{g}\},$$

$G_n(\lambda)$ satisfies (15) because of (18).

We now determine $x_2, x_3, \ldots, x_n$ in such a way that the expression

$$\left| \frac{R_i \Phi(-i)}{\{|\Phi(-i)|^2\}^{\frac{1}{2}}} - h'_n \right|^2 = \left| \int_{-\infty}^{+\infty} \left(\frac{\Omega_1(\lambda)}{\lambda^2+1} - \frac{G_n(\lambda)}{\lambda - i} \right) d\{E(\lambda)\overset{0}{g}\} \right|^2$$

$$= \left| \int_{-\infty}^{+\infty} \frac{\Omega_1(\lambda) + \sum_{p=2}^{n} x_p \Omega_p(\lambda)}{\lambda^2+1} \, d\{E(\lambda)\overset{0}{g}\} \right|^2$$

$$= \int_{-\infty}^{+\infty} \frac{\left|\Omega_1(\lambda) + \sum_{p=2}^{n} x_p \Omega_p(\lambda)\right|^2}{(\lambda^2+1)^2} \, d\rho(\lambda) \tag{19}$$

is a minimum, which we denote by M_n. Since the M_n evidently form a non-increasing sequence of positive numbers, $\lim_{n\to\infty} M_n$ exists. Thus the problem of approximating to the expression (17) is reduced to the question whether $\lim_{n\to\infty} M_n = 0$.

11. This, however, is answered by

LEMMA 1. *Let* $(Ax, x) = \sum_{p,q=1}^{\infty} a_{pq} \bar{x}_p x_q$ *be a positive definite bounded Hermitian form,* $(A^{(n)}x, x) = \sum_{p,q=1}^{n} a_{pq} \bar{x}_p x_q$ *the nth reduced form, and* $S(z) = \{s_{pq}(z)\} = (A - zI)^{-1}$ *the resolving matrix. If* M_n *denotes the minimum of* $(A^{(n)}x, x)$ *in the* $(n-1)$*-dimensional flat space* $x_1 = 1$*, then* $\lim_{n\to\infty} M_n = 0$ *if and only if* $\lim_{z\to 0} s_{11}(z) = \infty$*, z taking negative values only.*

Proof. If $S^{(n)}(z) = (A^{(n)} - zI^{(n)})^{-1} = \{s_{pq}^{(n)}(z)\}$,
an elementary calculation leads to $1/M_n = s_{11}^{(n)}(0)$. It is readily shown that for negative values of z,

$$\lim_{n\to\infty} s_{11}^{(n)}(z) = s_{11}(z), \qquad s_{11}^{(n)}(z) < s_{11}^{(n+p)}(z),$$

$$s_{11}^{(n)}(z) < s_{11}^{(n)}(z'), \qquad s_{11}(z) < s_{11}(z') \quad (z < z' \leqslant 0);$$

and this implies $\displaystyle \lim_{n\to\infty} M_n = \lim_{z\to 0} \frac{1}{s_{11}(z)}$,

which is the desired result.

12. In order to complete the proof of Theorem 6 we first notice that the expression (19) is the nth reduced form of a positive Hermitian form (Ax, x) with $x_1 = 1$,

$$a_{pq} = \int_{-\infty}^{+\infty} \frac{\Omega_p(\lambda)\overline{\Omega_q(\lambda)}}{(\lambda^2+1)^2} \, d\rho(\lambda). \tag{20}$$

If we apply to the integral (20) the substitution $\mu = \dfrac{1}{\lambda^2+1}$ and write

$$\psi_{1p}(\mu) = \Omega_p\left(\sqrt{\frac{1-\mu}{\mu}}\right), \qquad \psi_{2p}(\mu) = \Omega_p\left(-\sqrt{\frac{1-\mu}{\mu}}\right),$$

$$\omega_1(\mu) = -\int_0^\mu \mu\, d\rho\left(\sqrt{\frac{1-\mu}{\mu}}\right), \qquad \omega_2(\mu) = \int_0^\mu \mu\, d\rho\left(-\sqrt{\frac{1-\mu}{\mu}}\right),$$

we obtain

$$a_{pq} = \int_0^1 \mu\{\psi_{1p}(\mu)\overline{\psi_{1q}(\mu)}\, d\omega_1(\mu)+\psi_{2p}(\mu)\overline{\psi_{2q}(\mu)}\, d\omega_2(\mu)\}. \tag{21}$$

Since the $\Omega_p(\lambda)$ are supposed to form a complete ortho-normal system with respect to $d\rho(\lambda)/(\lambda^2+1)$, it is readily seen that (21) gives the spectral representation of the Hermitian form (Ax, x).

This implies further that the resolvent $s_{pq}(z)$ of the Hermitian form whose coefficients a_{pq} are defined by (20) or (21) is given by the equation

$$s_{pq}(z) = \int_0^1 \frac{1}{\mu-z}\{\psi_{1p}(\mu)\overline{\psi_{1q}(\mu)}\, d\omega_1(\mu)+\psi_{2p}(\mu)\overline{\psi_{2q}(\mu)}\, d\omega_2(\mu)\}.$$

If we again substitute $\mu = (\lambda^2+1)^{-1}$, we obtain

$$s_{pq}(z) = \int_{-\infty}^{+\infty} \frac{\Omega_p(\lambda)\Omega_q(\lambda)}{1-z(\lambda^2+1)}\, d\rho(\lambda). \tag{22}$$

Hence we see that the condition

$$\lim_{z\to 0} s_{11}(z) = \lim_{z\to 0} \int_{-\infty}^{+\infty} \frac{|\Omega_1(\lambda)|^2}{1-z(\lambda^2+1)}\, d\rho(\lambda) = \infty \quad (z<0)$$

and condition (iii) of Theorem 6 coincide.

Finally, the assertion (10) follows from the equation

$$\{H_{\bar{x}}\, h', \Phi(x)\} = 0,$$

which holds for every h' of $\mathfrak{D}$ and can readily be derived from (9). This completes the proof of Theorem 6.

13. Theorem 7. *Let $\overset{0}{H}$, R_x, $E(\lambda)$, $\overset{0}{g}$, and $\rho(\lambda)$ have the same meaning as in Theorem 6. Then, in order to construct all the c.H. prime transformations H of d.i. (m, m) to which $\overset{0}{H}$ belongs as a self-adjoint*

extension, we must consider all the sets of m linearly independent functions $\Omega_\mu(\lambda)$ $(\mu = 1, 2,\ldots, m)$ *which fulfil the following three conditions:*

(i) *the set of all points* λ *for which the system of m equations* $\Omega_\mu(\lambda) = 0$ *is satisfied is of* ρ*-measure zero;*

(ii) *the m integrals* $\displaystyle\int_{-\infty}^{+\infty} \frac{|\Omega_\mu(\lambda)|^2}{\lambda^2+1}\,d\rho(\lambda)$ *converge;*

(iii) *the integral* $\displaystyle\int_{-\infty}^{+\infty} \Big|\sum_{\mu=1}^{m} u_\mu \Omega_\mu(\lambda)\Big|^2\,d\rho(\lambda)$ *diverges for every system of m real or complex values of* u_μ.

Further, if we put

$$\Phi_\mu(x) = \int_{-\infty}^{+\infty} \frac{\Omega_\mu(\lambda)}{\lambda - x}\,d\{E(\lambda)\overset{0}{g}\} \quad (\mu = 1, 2,\ldots, m), \tag{23}$$

where x is any non-real number, we obtain the domain $\mathfrak{D}$ *which determines the desired transformation H of d.i.* (m, m) *by applying the transformation* R_i *to the linear manifold of all the elements of* $\mathfrak{H}$ *which are orthogonal to each of the m elements* $\Phi_\mu(-i)$. *We have finally* $H_x^*\Phi_\mu(x) = 0$ *for every* μ *and every non-real x.*

14. The proof follows the same lines as that of Theorem 6.

We first orthogonalize the m elements $\Phi_\mu(i)$ defined in (23) by E. Schmidt's familiar procedure in such a way that the equations (18) are satisfied for $1 \leqslant p,\ q \leqslant m$. With this the linear manifold spanned by the m elements $\Phi_\mu(i)$ remains unaltered. We further construct the other functions $\Omega_p(\lambda)$ $(p = m+1,\ m+2,\ldots)$ so that equations (18) hold for every p and q, and so that we obtain a complete ortho-normal set of functions $\Omega_p(\lambda)$ with respect to $\dfrac{d\rho(\lambda)}{\lambda^2+1}$.

The same argument as in § 10 leads to the expressions corresponding to (19), namely

$$|R_i\Phi_\mu(-i) - h_n'|^2 = \int_{-\infty}^{+\infty} \frac{\Big|\Omega_\mu(\lambda) + \sum\limits_{p=m+1}^{n} x_p\Omega_p(\lambda)\Big|^2}{(\lambda^2+1)^2}\,d\rho(\lambda)$$
$$(\mu = 1, 2,\ldots, m;\ n \geqslant m+1)$$

whose minima M_n^μ are now to be considered.

15. The question whether $\lim_{n\to\infty} M_n^\mu = 0$ is answered by

LEMMA 2. *Let (Ax,x), $(A^{(n)}x,x)$, and $S(z)$ have the same meaning as in Lemma 1. If, for $n \geqslant m+1$, M_n^* denotes the minimum of $(A^{(n)}x,x)$ in the $(n-m)$-dimensional flat space*

$$x_1 = 1, \qquad x_2 = x_3 = \dots = x_m = 0, \tag{24}$$

then $\lim_{n\to\infty} M_n^* = 0$ if

$$\lim_{z\to 0} \sum_{\mu,\nu=1}^{m} s_{\mu\nu}(z)\bar{u}_\mu u_\nu = \infty \quad (z<0) \tag{25}$$

for every system of complete numbers u_μ for which

$$\sum_{\mu=1}^{m} |u_\mu|^2 = 1. \tag{26}$$

Proof. Let $M_n(u_1, u_2, \dots, u_m) = M_n(u)$ denote the minimum of $(A^{(n)}x,x)$ in the $(n-1)$-dimensional flat space

$$\sum_{\mu=1}^{m} u_\mu x_\mu = 1, \tag{27}$$

where the u_μ satisfy (26). Then Lemma 1 implies that $\lim_{n\to\infty} M_n(u) = 0$ if and only if (25) is satisfied. This can readily be shown by subjecting $(A^{(n)}x,x)$ to a unitary transformation which carries the flat space (27) into $x_1 = 1$. We now consider the maximum μ_n of $M_n(u)$ on the unit sphere (26) and prove by a familiar argument that $\lim_{n\to\infty} \mu_n$ is also zero if (25) is satisfied. This implies that

$$M_n(u) < \epsilon \quad \text{for } n \geqslant N(\epsilon) \tag{28}$$

uniformly for any u on the unit sphere (26).

If M_n^* has the meaning defined in Lemma 2, the ellipsoid

$$(A^{(n)}x,x) = M_n^*$$

touches the flat space (24) in the point at which the minimum is attained. Hence the $(n-1)$-dimensional tangent plane to the ellipsoid at this point must contain the $(n-m)$-dimensional flat space (24). Hence its equation has the form

$$\sum_{\mu=1}^{m} u_\mu x_\mu = u_1, \tag{29}$$

where the u_μ satisfy (26). This implies, however, that M_n^* is also the minimum of $(A^{(n)}x,x)$ in the $(n-1)$-dimensional flat space (29). Moreover, since $|u_1| \leqslant 1$, because of (26), we have

$$M_n^* \leqslant M_n(u)$$

for the special system of values u_μ defined by (29). From this and (28) we obtain $\lim_{n\to\infty} M_n^* = 0$, the desired result.

16. In order to complete the proof of Theorem 7 we again have to consider the resolvent $S(z) = s_{pq}(z)$ of the Hermitian form (Ax, x) whose coefficients are defined by (20). Then the $s_{pq}(z)$ are determined by (22), and we obtain

$$\sum_{\mu,\nu=1}^{m} s_{\mu\nu}(z)\bar{u}_\mu u_\nu = \int_{-\infty}^{+\infty} \frac{\left|\sum_{\mu=1}^{m} u_\mu \Omega_\mu(\lambda)\right|^2}{1-z(\lambda^2+1)}\, d\rho(\lambda).$$

Hence we see that conditions (25) and (iii) of Theorem 7 coincide.

Finally, the assertion $H_x^*\Phi_\mu(x) = 0$ $(\mu = 1, 2,..., m)$ may be proved by the argument used in the last paragraph of § 12. This completes the proof of Theorem 7.

17. It may be remarked that a theorem similar to Theorems 6 and 7 also holds when the spectrum of $\overset{0}{H}$ is not simple but of multiplicity k, where k is a finite integer. We notice, however, that in this case we always have $m \geqslant k$ for every c.H. prime transformation H of d.i. (m, m) having $\overset{0}{H}$ as one of its self-adjoint extensions.

18. The application of the above methods to some special cases will be dealt with in a later note. This will give explicit expressions for the resolvents of all the self-adjoint extensions of a matrix $J^{(m)}$ of d.i. (m, m) which is composed in a suitable way of m Jacobi† matrices J_μ, each of d.i. $(1, 1)$. In this case the hypothesis of Theorem 5 will be satisfied.

† See (2) 530–614, especially 545, Theorem 10.27.

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